

# Excitation Spectra of Spin Models constructed from Quantized Affine Algebras of type $B_n^{(1)}$ , $D_n^{(1)}$

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**Abstract.** The energy and momentum spectrum of the spin models constructed from the vector representation of the quantized affine algebras of type  $B_n^{(1)}$  and  $D_n^{(1)}$  are computed using the approach of Davies et al. [1]. The results are for the anti-ferromagnetic (massive) regime, and they agree with the mass spectrum found from the factorized S-matrix theory by Ogievetsky et al. [2]. The other new result is the explicit realization of the fusion construction for the quantized affine algebras of type  $B_n^{(1)}$  and  $D_n^{(1)}$ .

## 1. Introduction

In [1] was given a new scheme for solving the six-vertex model and associated XXZ chain, in the antiferromagnetic regime, using the newly discovered quantum affine symmetry of the system. The approach of that paper has been extended to higher spin chains [3], to the higher rank case [4] and to the ABF models of Andrews, Baxter and Forrester [5]. All of these papers are concerned with models constructed on the (quantised affine) algebra  $A_n^{(1)}$ . In this paper we treat the case that the algebra is either  $B_n^{(1)}$  or  $D_n^{(1)}$ . (The bosonization of level 1 vertex operators for  $D_n^{(1)}$  and an integral formula for the correlation functions of the vertex model are given in [6].) Our principal result is to identify the “particle spectrum”, including their excitation energies and dispersion relations. We shall see also that the mass spectrum of these particles coincides with the mass spectrum found by Ogievetsky, Reshetikhin and Wiegmann from the factorised S-matrix theory [2].

The ideas are fundamentally different from the long established method of the Bethe Ansatz. One of the questions on Feynman’s last blackboards reads “describe centre of string without the ends” [7]. In the Bethe Ansatz approach, the answer is to join the ends and then take the infinite limit. With the scheme first presented in [1], the system is always infinite: one selects an arbitrary point as the centre and treats the system as the union of its left and right hand parts. The role played by the ends is then simply to select the various ground states.

To assist in the presentation of this paper, which has many technical details, we first make some general introductory comments in the present section, using the example of the XXZ chain. Then, in section 2, we discuss more specifically the application of the method to the problem at hand and state the principal results.

**1.1 Role of corner transfer matrices.** Corner transfer matrices (CTMs) were invented by Baxter [8,9], and proved to be an effective method for the evaluation of one-point functions in exactly solved lattice models of statistical mechanics [10–13]. CTMs are defined as a suitably normalised partition function of a whole quadrant of the lattice, so they are a transfer matrix which acts on a semi-infinite spin chain. From an early stage it was known that they have the following remarkably simple properties:

- (i) the spectrum is integer powers of a single parameter;
- (ii) they are exactly an exponential  $A(u) = \exp(uK)$ , where  $K$  is a spin-chain operator of the general form  $K = \sum_{l=1}^{\infty} lH_l$ , and  $u$  is the (additive) spectral parameter;
- (iii) the relation to more usual Hamiltonian  $H$  associated with the row transfer matrix  $T(u)$  is  $H = \sum_{l=-\infty}^{\infty} H_l$ ;
- (iv) this relation is manifest as a “boost property”  $T(u)A(v) = A(v)T(u+v)$  [14,15].

Since the transfer matrix  $T(u)$  takes a simple form (typically a shift operator) when  $u = 0$ , one sees that a full description of the row transfer matrix, and the Hamiltonian  $H$ , may be had if one can understand the eigenstates of CTMs and can also construct the shift operator in that representation. The vital key is the connection with quantum affine algebras [16–18]. In this picture, the six-vertex CTM generator  $K$ , acting on the left-hand semi-infinite part, is identified with the derivation operator  $d$  of the quantised affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$ , and its eigenstates with the weight vectors

of the level 1 modules  $V(\Lambda_0)$ ,  $V(\Lambda_1)$ . The two highest weights  $\Lambda_i$  correspond to the two possible ground states for the XXZ model. The eigenstates of the right hand part is the dual (a level  $-1$  module); so a suitable representation for the space of states of the entire chain is the direct sum of tensor products  $\bigoplus_{i,j=0,1} V(\Lambda_i) \otimes V(\Lambda_j)^{*a}$  — a level 0 module. (The antipode  $a$  must be used to construct the dual; the definitions are given below.) Unlike the irreducible level 1 modules, this representation is highly reducible: the most obvious reduction is a decomposition into  $n$ -particle states.

**1.2 Translation operator.** One may shift the selected point at which translational symmetry is broken, one site at a time, using the theory of quantum vertex operators (VOs) due to Frenkel and Reshetikhin [19]. This gives a viable representation-theoretic realisation of the translation operator  $T = T(0)$ . Since the derivation  $d$  was already identified with a Hamiltonian spin chain (the CTM) whose coefficients are linear in the position along the chain, the usual Hamiltonian spin chain becomes identified with a multiple of the operator  $(TdT^{-1} - d)$ , using the boost property.

VOs are algebra homomorphisms between highest weight modules  $V(\Lambda_i)$  ( $i = 0, 1$ ) and tensor products of the form  $V(\Lambda_{1-i}) \otimes V_z$  or  $V_z \otimes V(\Lambda_{1-i})$ . ( $V_z$  is a loop module, or affinisation of the usual spin-half  $U_q(\widehat{\mathfrak{sl}}_2)$  module.) For the two different orders of tensor product, we call the corresponding VOs type I and type II, respectively. A type I VO expands the eigenstates of the left-hand semi-infinite part in terms of the state of one local site and the eigenstates of a new semi-infinite part, truncated by that site. This “splits off” one site at the centre of the string by mapping  $V(\Lambda_i)$  to  $V(\Lambda_{1-i}) \otimes V_z$ . By using duality properties of modules and homomorphisms between them, there is another type I VO which adds this single site to the right hand semi-infinite part, that is, which maps  $V_z \otimes V(\Lambda_i)^{*a}$  to  $V(\Lambda_{1-i})^{*a}$ . The middle of the string is thereby translated by one lattice unit, whilst the sectors are switched corresponding to the translation. The advantage of this construction is that all expansions are convergent in  $q$  and  $z$ , in some domain of analyticity, even though they have rather complicated forms. Moreover, closed form expressions may be obtained for physical quantities using the powerful machinery of modern algebra and representation theory.

**1.3 Creation and annihilation operators.** Consider the ground states. Since the XXZ chain is  $U_q(\widehat{\mathfrak{sl}}_2)$  invariant in this infinite limit, the ground states  $\Psi_i$  ( $i = 0, 1$ ) must be unique, that is,  $\Psi_i$  must span a one-dimensional (and therefore trivial) subrepresentation. Using the canonical identifications

$$V(\Lambda_i) \otimes V(\Lambda_i)^{*a} \xrightarrow{\sim} \text{Hom}_{\mathbb{C}}(V(\Lambda_i), V(\Lambda_i)),$$

we see that the ground states are simply identity maps of  $V(\Lambda_i)$  into itself. This gives a simple expansion for  $\Psi_i$  in terms of any dual basis system for  $V(\Lambda_i)$  and  $V(\Lambda_i)^{*a}$ .

To create particles from a ground state is equivalent to finding maps  $\varphi_{\pm}^*(z)$  which create sub-modules in the various sectors  $V(\Lambda_i) \otimes V(\Lambda_j)^{*a}$ . For a single particle of spin  $1/2$  and momentum  $z = e^{ip}$ , the submodule must be isomorphic to the spin-half  $U_q(\widehat{\mathfrak{sl}}_2)$  module  $V_z$ , so we seek a homomorphism

$$\varphi_{\pm}^*(z) : V_z \otimes V(\Lambda_i) \xrightarrow{U_q} V(\Lambda_{1-i})$$

for the creation operator. Using the canonical identification

$$\mathrm{Hom}_{U_q}(L \otimes M, N) \xrightarrow{\sim} \mathrm{Hom}_{U_q}(M, L^{*a^{-1}} \otimes N)$$

one finds that the creation operators are equivalent to VOs of type II:

$$V(\Lambda_i) \xrightarrow{U_q} V_z^{*a^{-1}} \otimes V(\Lambda_{1-i}).$$

For the annihilation operators, we seek the following VOs of type II:

$$V(\Lambda_i) \xrightarrow{U_q} V_z \otimes V(\Lambda_{1-i}).$$

**1.4 Selection rules.** For the XXZ model, and also the higher spin [3] and higher rank [4] cases, the excitation energies, particle spectra and dispersion relations are compared with earlier Bethe Ansatz calculations. These are important checks on the validity of the representation-theoretic picture versus the physical content. But it is equally important to note that the information, including all necessary commutation relations, scalar factors and selection rules, may be obtained directly from representation theory. As a simple example, consider again the XXZ model. The question arises, why is there only a spin-half particle in the spectrum? The answer from representation theory is that, if one considers the tensor product  $V_z^s \otimes V(\Lambda_i)$ , where  $s$  is spin, there is no VO which maps this into the space  $V(\Lambda_j)$  except when  $s = 1/2$  [20]. The language of quantum VOs may be rather new to mathematical physics, but selection rules are certainly not. Similarly, for the higher rank case  $A_n^{(1)}$ , all the level 1 modules  $V(\Lambda_i)$ ,  $i = 0, \dots, n$ , correspond to spaces of states with differing boundary conditions. For this case, let  $V_z^{(k)}$ ,  $k = 2, \dots, n$ , be the  $k$ -th fusion of the (affinised) vector representation  $V_z^{(1)}$ ; then it is known that there is a VO which maps each  $V_z^{(k)} \otimes V(\Lambda_i)$  into a unique  $V(\Lambda_j)$ , so that spectrum has a total of  $n$  different particle types.

## 2. Main Results

**2.1 Hamiltonian and the anti-ferromagnetic regime.** We consider affine Lie algebras  $\mathfrak{g} = B_n^{(1)}$ , ( $n \geq 3$ ) (resp.  $D_n^{(1)}$ , ( $n \geq 4$ )). Their Dynkin diagrams are shown below.

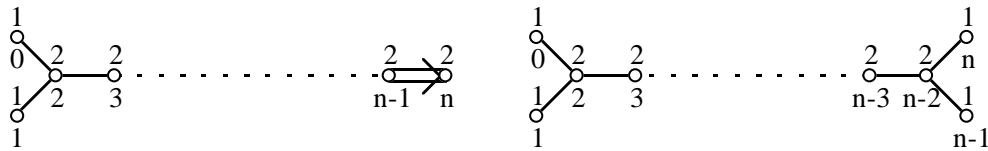


Figure 1: Dynkin diagrams of  $B_n^{(1)}$  and  $D_n^{(1)}$

Let  $U_q(\mathfrak{g})$  be the quantum affine algebra associated with  $\mathfrak{g}$ . Consider the  $R$ -matrix  $R(z) = \overline{R}^{(1,1)}(z)$  (3.12) for the vector representation of  $U_q(\mathfrak{g})$ .  $R(z)$  acts on the vector space  $V \otimes V$ , where  $\dim V = 2n + 1$  (resp.  $2n$ ). Given the  $R$ -matrix, it provides the Boltzmann weights of a vertex model with  $(2n + 1)$  (resp.  $2n$ ) states, satisfying the Yang-Baxter equation (3.7). Set

$$\begin{aligned} h &= -(q^t - q^{-t}) \cdot z \frac{d}{dz} \log PR(z) \Big|_{z=1}, \\ t &= 1/2 \text{ } (\mathfrak{g} = B_n^{(1)}), \quad t = 1 \text{ } (\mathfrak{g} = D_n^{(1)}). \end{aligned} \quad (2.1)$$

Here  $P$  denotes the transposition:  $Pu \otimes v = v \otimes u$ . Define the Hamiltonian

$$\mathcal{H} = \sum_{l \in \mathbf{Z}} h_{l+1,l}$$

acting on the infinite tensor product

$$\cdots \otimes_{l+1} V \otimes_l V \otimes_{l-1} V \otimes \cdots.$$

Here  $h_{l+1,l}$  acts as  $h$  on the  $(l + 1)$ -th and  $l$ -th components and as the identity on the other components.

The structure of space of states (physical regime) depends on the region of the parameter  $q$  in the Hamiltonian. Unlike the Bethe Ansatz method, the present approach is effective only for a particular region of  $q$ , corresponding to the anti-ferromagnetic or massive regime. It is in this regime that the necessary infinite limit of the physical system may be defined in such a way that the CTM becomes identified with the derivation of the quantum affine algebra. Because of the large freedom of gauge transformations of the Hamiltonian, we do not attempt herein to investigate this problem along the lines of [16,17], although we hope to do so in a future publication. (In fact, even the general  $A_n^{(1)}$  case has not been analysed in such a way, although the agreement with existing Bethe Ansatz results provides a strong check in that case.) Here we just state our conjecture, which is a simple extension of known results, and takes account of the isomorphisms between the  $A_1^{(1)}$  and  $B_1^{(1)}$  case and the  $A_3^{(1)}$  and  $D_3^{(1)}$  case. It is

$$-1 < q^t < 0,$$

where  $t$  is given in (2.1). The prefactor  $-(q^t - q^{-t})$  of  $h$  is chosen to conform to conventions set in previous work which relates quantum affine algebras and CTMs. With this choice, it is negative.

**2.2 Physical sectors and the space of states.** Physical sectors are the eigenstates of the CTM with some appropriate boundary condition. Candidates for them are integrable highest weight modules. The situation here is quite different from the  $A_n^{(1)}$  case. There, all the highest weight representations corresponding to the fundamental weights  $V(\Lambda_i)$ ,  $i = 0, \dots, n$  are of level 1. Since the affinised modules used to construct

translation and creation operators are of level 0, all of the fundamental modules are involved in the physics, whilst all of the higher level modules are excluded. For the orthogonal algebras, the fundamental modules  $V(\Lambda_0)$ ,  $V(\Lambda_1)$ ,  $V(\Lambda_n)$  (also  $V(\Lambda_{n-1})$  for  $D_n^{(1)}$ ) are level 1 whilst all others are level 2. For the anti-ferromagnetic regime to which we restrict our attention, there are therefore three (resp. four) CTM ground state sectors, corresponding to these level-1 modules. Therefore, the whole physical space of states may be identified with

$$\bigoplus_{i,j} V(\Lambda_i) \otimes V(\Lambda_j)^{*a},$$

where the sum is over  $0, 1, n$  (also  $n-1$  for  $D_n^{(1)}$ ). As a  $U_q(\mathfrak{g})$ -module, the sector  $V(\Lambda_i) \otimes V(\Lambda_j)^{*a}$  is canonically equivalent to the space of linear maps  $V(\Lambda_j) \xrightarrow{\mathbb{C}} V(\Lambda_i)$ , and the ground states of the infinite chain are the identity maps which require  $i = j$ .

**2.3 Selection rules.** Let  $V_z^{(1)}$  be the affinised vector representation of  $B_n^{(1)}$  (resp.  $D_n^{(1)}$ ), and let  $V_z^{(k)}$  be the affinised  $k$ 'th fusion, with  $k = 2, \dots, n-1$  (resp.  $k = 2, \dots, n-2$ ). We shall also need to consider the affinised spin representations, which we denote here as  $V_z^{(n)}$  (also  $V_z^{(n-1)}$  for  $D_n^{(1)}$ ). To each of the representations  $V_z^{(k)}$  there correspond type II VOs between physical sectors, and therefore massive excitations of the system for each  $1 \leq k \leq n$ . We shall describe the selection rules in detail. This will provide a proper setting for the excitation spectra formulae which follow, and give a comprehensive picture of how the various sectors are inter-related. First, for the vector representation  $V_z^{(1)}$  we have the following VOs (the two columns are for  $B_n^{(1)}$  and  $D_n^{(1)}$  as indicated):

$$\begin{array}{ll} B_n^{(1)} & D_n^{(1)} \\ V_z^{(1)} \otimes V(\Lambda_0) \longrightarrow V(\Lambda_1), & V_z^{(1)} \otimes V(\Lambda_0) \longrightarrow V(\Lambda_1), \\ V_z^{(1)} \otimes V(\Lambda_1) \longrightarrow V(\Lambda_0), & V_z^{(1)} \otimes V(\Lambda_1) \longrightarrow V(\Lambda_0), \\ V_z^{(1)} \otimes V(\Lambda_n) \longrightarrow V(\Lambda_n). & V_z^{(1)} \otimes V(\Lambda_n) \longrightarrow V(\Lambda_{n-1}), \\ & V_z^{(1)} \otimes V(\Lambda_{n-1}) \longrightarrow V(\Lambda_n). \end{array} \quad (2.2)$$

So, for example, the creation operator corresponding to  $V_z^{(1)}$ , acting on the ground state in the sector  $V(\Lambda_0) \otimes V(\Lambda_0)^{*a}$ , creates a massive excitation in the sector  $V(\Lambda_1) \otimes V(\Lambda_0)^{*a}$ , and acting on the ground state in sector  $V(\Lambda_n) \otimes V(\Lambda_n)^{*a}$  it creates an excitation in the sector  $V(\Lambda_n) \otimes V(\Lambda_n)^{*a}$  for  $B_n^{(1)}$  or  $V(\Lambda_{n-1}) \otimes V(\Lambda_n)^{*a}$  for  $D_n^{(1)}$ .

For the fusion representations, the selection rules are exactly the same as one would find by iterating the vector creation operator. This is consistent with the fact that  $V^{(k)}$  is projected out of  $(V^{(1)})^{\otimes k}$ . Thus, when  $k$  is odd, the picture is the same as (2.2), whereas when  $k$  is even we have

$$V_z^{(k)} \otimes V(\Lambda_i) \longrightarrow V(\Lambda_i) \quad \begin{cases} (i = 0, 1, n) & \text{for } B_n^{(1)}, \\ (i = 0, 1, n-1, n) & \text{for } D_n^{(1)}. \end{cases} \quad (2.3)$$

The remaining excitations come from the spin representations  $V_z^{(n)}$  (and  $V_z^{(n-1)}$  for  $D_n^{(1)}$ ): these are the only ones which can mix  $V(\Lambda_0)$  and  $V(\Lambda_1)$  with  $V(\Lambda_n)$  (and  $V(\Lambda_{n-1})$  for  $D_n^{(1)}$ ). The possibilities are:

$$\begin{array}{ll}
B_n^{(1)} & D_n^{(1)} \\
V_z^{(n)} \otimes V(\Lambda_j) \longrightarrow V(\Lambda_n), & V_z^{(n)} \otimes V(\Lambda_j) \longrightarrow V(\Lambda_{n-j}), \\
V_z^{(n)} \otimes V(\Lambda_n) \longrightarrow V(\Lambda_j), & V_z^{(n-1)} \otimes V(\Lambda_j) \longrightarrow V(\Lambda_{n+j-1}), \\
& V_z^{(n-i)} \otimes V(\Lambda_{n-j}) \longrightarrow V(\Lambda_j), \\
& V_z^{(n+i-1)} \otimes V(\Lambda_{n-j}) \longrightarrow V(\Lambda_{1-j}),
\end{array} \tag{2.4}$$

where  $j$  takes the values 0, 1, and in the  $D_n^{(1)}$  case  $i = 0$  (resp.  $i = 1$ ) if  $n$  is even (resp.  $n$  is odd).

**2.4 Creation/annihilation operators and the transfer matrix.** As we have stated in the introduction, type II VOs are used for the mathematical formulations of creation and annihilation operators. Let us denote them by  $\varphi_{\lambda,I}^{(k)*}(z)$  and  $\varphi_{\lambda,I}^{(k)}(z)$ . Here  $k$  is related to the affinised representation  $V_z^{(k)}$  we specify for the type II VO.  $\lambda$  is the ground state sector they act,  $I$  is an index of base vectors of  $V_z^{(k)}$ , and  $z$  is the momentum. See subsection 7.3 for the precise definition. For a level 1 dominant integral weight  $\lambda$ , *i.e.*  $\lambda = \Lambda_i$  with  $i = 0, 1, n$  (and  $n-1$  for  $D_n^{(1)}$ ), define  $\lambda^{(k)*}$ ,  $\lambda^{(k)}$  ( $k = 1, \dots, n$ ) to be the one determined uniquely from the following selection rules:

$$\begin{aligned}
V_z^{(k)} \otimes V(\lambda) &\longrightarrow V(\lambda^{(k)*}), \\
V(\lambda) &\longrightarrow V_z^{(k)} \otimes V(\lambda^{(k)}).
\end{aligned}$$

We can see  $(\lambda^{(k)})^{(k)*} = (\lambda^{(k)*})^{(k)} = \lambda$ .  $\varphi_{\lambda,I}^{(k)*}(z)$  and  $\varphi_{\lambda,I}^{(k)}(z)$  are operators from  $V(\lambda)$  to  $V(\lambda^{(k)*})$  and  $V(\lambda^{(k)})$ , respectively.

The row transfer matrix can also be formulated via VO of type I (See subsection 7.3):

$$T_{\lambda\mu}^{\lambda^{(1)}\mu^{(1)}}(z) : V(\lambda) \otimes V(\mu)^{*a} \longrightarrow V(\lambda^{(1)}) \otimes V(\mu^{(1)})^{*a}.$$

From the commutation relations of VOs, we can derive those between the transfer matrix and creation (annihilation) operators. They are given by

$$\begin{aligned}
\varphi_{\lambda^{(1)},I}^{(k)*}(z_1) T_{\lambda\mu}^{\lambda^{(1)}\mu^{(1)}}(z_2) &= \tau^{(k)}(z_1/z_2) T_{\lambda^{(k)*}\mu}^{(\lambda^{(k)*})^{(1)}\mu^{(1)}}(z_2) \varphi_{\lambda,I}^{(k)*}(z_1), \\
\varphi_{\lambda^{(1)},I}^{(k)}(z_1) T_{\lambda\mu}^{\lambda^{(1)}\mu^{(1)}}(z_2) &= \tau^{(k)}(z_1/z_2)^{-1} T_{\lambda^{(k)}\mu}^{(\lambda^{(k)})^{(1)}\mu^{(1)}}(z_2) \varphi_{\lambda,I}^{(k)}(z_1).
\end{aligned}$$

Here  $\tau^{(k)}(z)$  is a scalar function defined in the next subsection.

**2.5 Excitation spectra.** Because of the number of possible ground state sectors, there is considerable detail in the selection rules for massive particle excitations. By contrast, the excitation spectra are quite simple. Regardless of the sectors, there is just a

single dispersion relation for each value of  $1 \leq k \leq n$ : moreover there is no distinction between  $k = n - 1$  and  $k = n$  in the  $D_n^{(1)}$  case. Here we discuss briefly the formulae for the  $A_n^{(1)}$ ,  $B_n^{(1)}$  and  $D_n^{(1)}$  cases, and compare with the mass spectrum found by Ogievetsky, Reshetikhin and Wiegmann from the factorised S-matrix theory [2]. We note here that Nakanishi [21] gave a systematic way to obtain the mass spectrum via the Yangian algebra symmetry.

Let us first discuss the  $A_n^{(1)}$  case, which was treated by Date and Okado [4] using the quantum affine symmetry. It was shown there that the excitation spectrum is given as

$$e^{-ip^{(k)}(\theta)} = \tau^{(k)}(z), \quad \epsilon^{(k)}(\theta) = -(q - q^{-1})z \frac{d}{dz} \log \tau^{(k)}(z), \quad (2.5)$$

where  $\epsilon^{(k)}(\theta)$ ,  $p^{(k)}(\theta)$  are the energy and momentum with rapidity variable  $\theta$  ( $-z = e^{2i\theta}$ ), and

$$\tau^{(k)}(z) = z^{\frac{k}{n+1}-1} \frac{\Theta_{q^{2n+2}}(-(-q)^k z)}{\Theta_{q^{2n+2}}(-(-q)^k z^{-1})}, \quad (2.6)$$

$$\Theta_p(z) = (z; p)_\infty (pz^{-1}; p)_\infty (p; p)_\infty, \quad (2.7)$$

$$(z; p)_\infty = \prod_{j=0}^{\infty} (1 - p^j z). \quad (2.8)$$

Note that the affine Lie algebra of type  $A_n^{(1)}$  is  $\widehat{\mathfrak{sl}}_{n+1}$ , so that  $n$  in [4] should be shifted by 1.

Date and Okado also show that this is exactly the result previously obtained by Babelon, deVega and Viallet [22] from the Bethe Ansatz. The latter also consider the scaling limit of small lattice spacing and small rapidity, to obtain the relativistic spectrum

$$P^{(k)}(\theta) = \mu \sin\left(\frac{\pi k}{n+1}\right) \text{sh } v, \quad E^{(k)}(\theta) = \mu \sin\left(\frac{\pi k}{n+1}\right) \text{ch } v.$$

Here  $P^{(k)}$ ,  $E^{(k)}$  and  $v$  are appropriately scaled version of  $p^{(k)}$ ,  $\epsilon^{(k)}$  and  $\theta$ . The mass spectrum can be read off directly from this:

$$m^{(k)} = \mu \sin\left(\frac{\pi k}{n+1}\right), \quad k = 1, \dots, n.$$

This agrees with the result obtained in [2,21].

For the present calculations, we find herein that the excitation spectrum is given via the following functions, in the cases that the excitation arises from the fusion or spin representation:

$$\tau^{(k)}(z) = z^{-1} \frac{\Theta_{\xi^2}(-(-q)^k z) \Theta_{\xi^2}(-(-q)^{-k} \xi z)}{\Theta_{\xi^2}(-(-q)^k z^{-1}) \Theta_{\xi^2}(-(-q)^{-k} \xi z^{-1})}, \quad (\text{fusion})$$

$$= z^{-1/2} \frac{\Theta_{\xi^2}(-s \xi^{1/2} z)}{\Theta_{\xi^2}(-s \xi^{1/2} z^{-1})}. \quad (\text{spin})$$



Here  $\xi = q^{h^\vee}$  where  $h^\vee$  is the dual Coxeter number of  $\mathfrak{g}$  ( $2n - 1$  for  $B_n^{(1)}$ ,  $2n - 2$  for  $D_n^{(1)}$ ), and  $s = (-)^n$  ( $\mathfrak{g} = B_n^{(1)}$ ),  $s = (-)^{n-1}$  ( $\mathfrak{g} = D_n^{(1)}$ ). For the fusion excitations,  $k$  takes the values  $1 \leq k \leq n - 1$  ( $\mathfrak{g} = B_n^{(1)}$ ) or  $1 \leq k \leq n - 2$  ( $\mathfrak{g} = D_n^{(1)}$ ) ( $k = 1$  is the vector representation); the remaining values of  $k$  are for the spin excitations. (The authors could not find any published Bethe Ansatz result to compare with. But J. Suzuki sent M.O. a fax showing that, apart from overall sign, our results agree with those computed from their Bethe Ansatz data in [23].) If we apply a similar scaling limit as in the  $A_n^{(1)}$  case, we obtain

fusion	spin
$P^{(k)}(\theta) = 2\mu \sin\left(\frac{\pi k}{h^\vee}\right) \text{sh } v,$	$P^{(k)}(\theta) = \mu \text{sh } v,$
$E^{(k)}(\theta) = 2\mu \sin\left(\frac{\pi k}{h^\vee}\right) \text{ch } v.$	$E^{(k)}(\theta) = \mu \text{ch } v.$

The mass spectrum agrees with that obtained in [2,21]. An interesting feature is the factor 2 which occurs for the fusion excitations; this comes from the double  $\Theta_{\xi^2}$  factors in  $\tau^{(k)}(z)$ .

The excitation spectrum is the main physical result of this paper. However, the remaining sections, and the appendix, which are devoted to the derivation of these results, contain a considerable amount of new information about the  $R$ -matrices for the quantised affine algebras of type  $B_n^{(1)}$  and  $D_n^{(1)}$ . In particular, the fusion construction is treated in some detail.

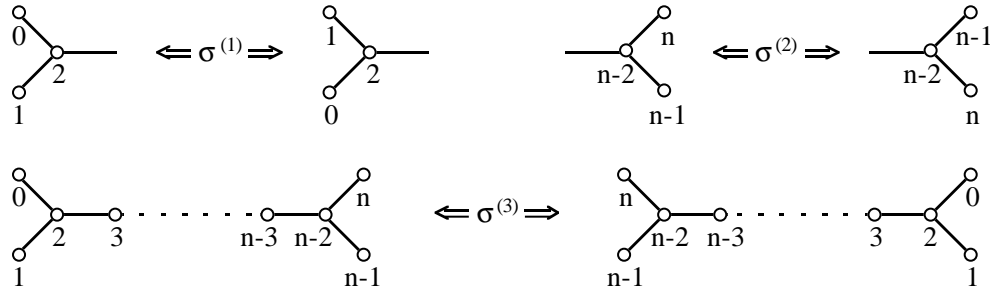
### 3. Fundamental representations and $R$ -matrices

**3.1 Quantum affine algebras.** We consider affine Lie algebras  $\mathfrak{g} = B_n^{(1)}$  ( $n \geq 3$ ),  $D_n^{(1)}$  ( $n \geq 4$ ). Let  $\mathfrak{h}^*, \Lambda_i, h_i = \alpha_i^\vee, \alpha_i, \delta = \sum_{i=0}^n a_i \alpha_i$  and  $d$  have the same meaning as in [24]. Their Dynkin diagrams are shown in Figure 1. The lower (resp. upper) integer of each vertex stands for the index  $i$  (resp. the value  $a_i$ ). The dual Coxeter number is defined as  $h^\vee = \sum_{i=0}^n a_i^\vee$ , where  $a_i^\vee = a_i$  except that for  $\mathfrak{g} = B_n^{(1)}$   $a_n^\vee = a_n/2$ , so that we have  $h^\vee = 2n - 1$  ( $\mathfrak{g} = B_n^{(1)}$ ),  $2n - 2$  ( $\mathfrak{g} = D_n^{(1)}$ ). The invariant bilinear form  $(|)$  in [24] is so normalised that  $(\theta|\theta) = 2$ , where  $\theta = \delta - \alpha_0$ . We put  $\rho = \sum_{i=0}^n \Lambda_i$ . For  $\lambda \in \mathfrak{h}^*$   $\bar{\lambda}$  denotes the orthogonal projection of  $\lambda$  on  $\bar{\mathfrak{h}}^*$ , where  $\bar{\mathfrak{h}}^*$  is the linear span of the classical roots  $\alpha_1, \dots, \alpha_n$ . Following [25] we introduce an orthonormal basis  $\{\epsilon_1, \dots, \epsilon_n\}$  of  $\bar{\mathfrak{h}}^*$ , by which  $\alpha_i, \bar{\Lambda}_i, \bar{\rho}$  are represented below.

$$\begin{aligned}
\alpha_i &= \epsilon_i - \epsilon_{i+1} & (1 \leq i \leq n-1), \\
&= \epsilon_n & (i = n \text{ for } B_n^{(1)}), \\
&= \epsilon_{n-1} + \epsilon_n & (i = n \text{ for } D_n^{(1)}), \\
\bar{\Lambda}_i &= \epsilon_1 + \dots + \epsilon_i & (1 \leq i \leq n-1 \text{ for } B_n^{(1)}, 1 \leq i \leq n-2 \text{ for } D_n^{(1)}), \\
&= \frac{\epsilon_1 + \dots + \epsilon_{n-1} - \epsilon_n}{2} & (i = n-1 \text{ for } D_n^{(1)}),
\end{aligned}$$

$$\begin{aligned}
&= \frac{\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n}{2} \quad (i = n), \\
2\bar{\rho} &= (2n-1)\epsilon_1 + (2n-3)\epsilon_2 + \cdots + 3\epsilon_{n-1} + \epsilon_n \quad (\text{for } B_n^{(1)}), \\
&= (2n-2)\epsilon_1 + (2n-4)\epsilon_2 + \cdots + 2\epsilon_{n-1} \quad (\text{for } D_n^{(1)}).
\end{aligned}$$

For our affine Lie algebras there are Dynkin diagram automorphisms described in Figure 2. We shall use these symmetries to reduce the number of cases of two point functions which must be calculated.



**Figure 2:** Dynkin diagram automorphisms

The quantum affine algebra  $U_q(\mathfrak{g})$  is defined as the  $\mathbf{Q}(q)$ -algebra with 1 generated by  $e_i, f_i, t_i (i = 0, \dots, n), q^d$  satisfying

$$\begin{aligned}
[t, t'] &= 0 & \text{for } t, t' \in \{t_0, \dots, t_n, q^d\}, \\
t_i e_j t_i^{-1} &= q^{(\alpha_i | \alpha_j)} e_j, & t_i f_j t_i^{-1} &= q^{-(\alpha_i | \alpha_j)} f_j, \\
q^d e_j q^{-d} &= q^{\delta_{j0}} e_j, & q^d f_j q^{-d} &= q^{-\delta_{j0}} f_j, \\
[e_i, f_j] &= \delta_{ij} (t_i - t_i^{-1}) / (q_i - q_i^{-1}), \\
\sum_{k=0}^b (-)^k e_i^{(k)} e_j e_i^{(b-k)} &= \sum_{k=0}^b (-)^k f_i^{(k)} f_j f_i^{(b-k)} = 0 \quad (i \neq j),
\end{aligned}$$

where  $b = 1 - \langle h_i, \alpha_j \rangle$ . Here we have set  $q_i = q^{(\alpha_i | \alpha_i)/2}$ ,  $[m]_i = (q_i^m - q_i^{-m}) / (q_i - q_i^{-1})$ ,  $[k]_i! = \prod_{m=1}^k [m]_i$ ,  $e_i^{(k)} = e_i^k / [k]_i!$ ,  $f_i^{(k)} = f_i^k / [k]_i!$ . We denote by  $U'_q(\mathfrak{g})$  the subalgebra of  $U_q(\mathfrak{g})$  generated by the elements  $e_i, f_i, t_i (i = 0, \dots, n)$ . Let  $x_i$  be any of  $e_i, f_i, t_i$ . We define algebra automorphisms  $\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}$  of  $U'_q(\mathfrak{g})$  as follows (Note that  $\sigma^{(2)}, \sigma^{(3)}$  are only for  $D_n^{(1)}$ ):

$$\begin{aligned}
\sigma^{(1)}(x_i) &= x_{1-i} \quad (i = 0, 1), & \sigma^{(1)}(x_i) &= x_i \quad (i \geq 2), \\
\sigma^{(2)}(x_i) &= x_i \quad (i \leq n-2), & \sigma^{(2)}(x_{n-i}) &= x_{n+i-1} \quad (i = 0, 1), \\
\sigma^{(3)}(x_i) &= x_{n-i} \quad (\forall i).
\end{aligned} \tag{3.1}$$

For a representation  $(\pi, V)$  of  $U'_q(\mathfrak{g})$ , we put  $V_z = V \otimes \mathbf{Q}(q)[z, z^{-1}]$ , and lift  $\pi$  to a representation  $(\pi_z, V_z)$  of  $U_q(\mathfrak{g})$  as follows:

$$\begin{aligned}\pi_z(e_i)(v \otimes z^n) &= \pi(e_i)v \otimes z^{n+\delta_{i0}}, \\ \pi_z(f_i)(v \otimes z^n) &= \pi(f_i)v \otimes z^{n-\delta_{i0}}, \\ \pi_z(t_i)(v \otimes z^n) &= \pi(t_i)v \otimes z^n, \\ \pi_z(q^d)(v \otimes z^n) &= v \otimes (qz)^n.\end{aligned}\tag{3.2}$$

By abuse of notation we sometimes write  $V$  for  $V_z$  where the context is clear.

The coproduct  $\Delta$  and the antipode  $a$  are defined as follows.

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \tag{3.3}$$

$$\begin{aligned}\Delta(t_i) &= t_i \otimes t_i, & \Delta(q^d) &= q^d \otimes q^d, \\ a(e_i) &= -t_i^{-1}e_i, & a(f_i) &= -f_i t_i, & a(t_i) &= t_i^{-1}, & a(q^d) &= q^{-d}.\end{aligned}\tag{3.4}$$

**3.2 Fundamental representations and definition of the  $R$ -matrix.** To each  $k$  ( $1 \leq k \leq n$ ) we associate a fundamental representation of  $U'_q(\mathfrak{g})$  denoted by  $(\pi^{(k)}, V^{(k)})$ . If  $k = 1$ , we call it the vector representation. If  $k = n$  (resp.  $k = n - 1, n$ ), we call it the spin representation for  $U'_q(B_n^{(1)})$  (resp.  $U'_q(D_n^{(1)})$ ). The other fundamental representations are constructed via the fusion construction in the subsequent section. For each fundamental representation  $V^{(k)}$  we shall define linear maps  $\sigma^{(i)} : V^{(k)} \longrightarrow V^{(k')}$  ( $i = 1$  for  $B_n^{(1)}$ ,  $i = 1, 2, 3$  for  $D_n^{(1)}$ ) satisfying

$$\pi^{(k')}(\sigma^{(i)}(x)) \sigma^{(i)}(v) = \sigma^{(i)}(\pi^{(k)}(x)v) \quad (x \in U'_q(\mathfrak{g}), v \in V^{(k)}), \tag{3.5}$$

where  $k' = k$  except that  $V^{(n-1)}$  and  $V^{(n)}$  may be interchanged by  $\sigma^{(i)}$  for  $\mathfrak{g} = D_n^{(1)}$ . We use the same notation for the linear maps as for the algebra automorphisms, in the belief that this should assist the presentation rather than cause confusion. We shall consider the maps in detail as the various representations are presented.

Let  $(\pi^{V_i}, V_i)$  ( $i = 1, 2$ ) be arbitrary representations of  $U'_q(\mathfrak{g})$ . Let  $\overline{R}^{V_1 V_2}(z) \in \text{End}(V_1 \otimes V_2)$  be the  $R$ -matrix, which satisfies

$$\overline{R}^{V_1 V_2}(z_1/z_2)(\pi_{z_1}^{V_1} \otimes \pi_{z_2}^{V_2})\Delta(x) = (\pi_{z_1}^{V_1} \otimes \pi_{z_2}^{V_2})\Delta'(x)\overline{R}^{V_1 V_2}(z_1/z_2), \quad \forall x \in U_q(\mathfrak{g}), \tag{3.6}$$

where  $\Delta' = \sigma \circ \Delta$  and  $\sigma$  denotes the transposition. From each representation space we normally choose a reference vector  $u_i$  ( $i = 1, 2$ ) and normalise by

$$\overline{R}^{V_1 V_2}(z)u_1 \otimes u_2 = u_1 \otimes u_2.$$

Together with another representation  $(\pi^{V_3}, V_3)$  of  $U'_q(\mathfrak{g})$ , the  $R$ -matrices satisfy the Yang–Baxter equation:

$$\begin{aligned}\overline{R}^{V_1 V_2}(z_1/z_2)\overline{R}^{V_1 V_3}(z_1/z_3)\overline{R}^{V_2 V_3}(z_2/z_3) \\ = \overline{R}^{V_2 V_3}(z_2/z_3)\overline{R}^{V_1 V_3}(z_1/z_3)\overline{R}^{V_1 V_2}(z_1/z_2) \quad \text{on } V_1 \otimes V_2 \otimes V_3.\end{aligned}\tag{3.7}$$

Here  $\overline{R}^{V_i V_j}(z)$  acts on the third space  $V_k$  as the identity.

Let  $\mathcal{R}'(z)$  be the modified universal  $R$  defined in appendix 1 of [3]. Since our  $R$ -matrix is uniquely determined up to a scalar from the intertwining property (3.6), we have

$$(\pi^{V_1} \otimes \pi^{V_2})(\mathcal{R}'(z)) = \beta^{V_1 V_2}(z) \overline{R}^{V_1 V_2}(z).$$

In order to obtain two point functions by solving the  $q$ -KZ equation, we need to know the form of  $\beta^{V_1 V_2}(z)$ . For this purpose, we consider the second inversion relation

$$\alpha^{V_1 V_2}(z)((\overline{R}^{V_1 V_2}(z)^{-1})^{t_1})^{-1} t_1 = (q^{-2\overline{p}} \otimes \text{id}) \overline{R}^{V_1 V_2}(z \xi^{-2})(q^{2\overline{p}} \otimes \text{id}). \quad (3.8)$$

Here

$$\xi = q^{h^\vee}$$

and  $\alpha^{V_1 V_2}(z)$  is a scalar function. If we replace  $\overline{R}^{V_1 V_2}(z)$  by  $(\pi^{V_1} \otimes \pi^{V_2})(\mathcal{R}'(z))$  in (3.8) we know that  $\alpha^{V_1 V_2}(z) = 1$ . As is prescribed in section 4 of [19], this fact enables us to get a difference equation for  $\beta^{V_1 V_2}(z)$  and thereby to calculate such factors.

**3.3 Vector representations and their  $R$ -matrices.** Define an index set  $J$  by

$$\begin{aligned} J &= \{0, \pm 1, \dots, \pm n\}, & \text{for } \mathfrak{g} = B_n^{(1)}, \\ &= \{\pm 1, \dots, \pm n\}, & \text{for } \mathfrak{g} = D_n^{(1)}, \end{aligned}$$

and set  $N = |J|$ . We introduce a linear order  $\prec$  in  $J$  by

$$1 \prec 2 \prec \dots \prec n \prec 0 \prec -n \prec \dots \prec -2 \prec -1.$$

We shall also use the usual order  $<$  in  $J$ .

We shall define the vector representation  $(\pi^{(1)}, V^{(1)})$  of  $U'_q(\mathfrak{g})$ . It is the fundamental representation associated with the vertex 1 in the Dynkin diagram. The base vectors of  $V^{(1)}$  are given by  $\{v_j \mid j \in J\}$ . The weight of  $v_j$  is given by  $\epsilon_j (j > 0)$ ,  $-\epsilon_{-j} (j < 0)$ ,  $0 (j = 0)$ . We take  $v_1$  as a reference vector. Set

$$s = \begin{cases} (-)^n & \text{for } \mathfrak{g} = B_n^{(1)}, \\ (-)^{n-1} & \text{for } \mathfrak{g} = D_n^{(1)}. \end{cases} \quad (3.9)$$

Denoting the matrix units by  $E_{ij}$  i.e.  $E_{ij}v_k = \delta_{jk}v_i$ , the actions of the generators read as follows  $(\pi^{(1)}(f_i) = \pi^{(1)}(e_i)^t)$ :

$$\begin{aligned} \pi^{(1)}(e_0) &= s(E_{-1,2} - E_{-2,1}), \\ \pi^{(1)}(t_0) &= \sum_{j \in J} q^{-\delta_{j1} - \delta_{j2} + \delta_{j,-1} + \delta_{j,-2}} E_{jj}, \\ \pi^{(1)}(e_i) &= E_{i,i+1} - E_{-i-1,-i} \quad (1 \leq i \leq n-1), \\ \pi^{(1)}(t_i) &= \sum_{j \in J} q^{\delta_{ji} - \delta_{j,i+1} + \delta_{j,-i-1} - \delta_{j,-i}} E_{jj} \quad (1 \leq i \leq n-1), \end{aligned}$$

$$\begin{aligned}
\pi^{(1)}(e_n) &= \sqrt{[2]_n}(E_{n0} - E_{0,-n}) && \text{for } \mathfrak{g} = B_n^{(1)}, \\
&= E_{n-1,-n} - E_{n,-n+1} && \text{for } \mathfrak{g} = D_n^{(1)}, \\
\pi^{(1)}(t_n) &= \sum_{j \in J} q^{\delta_{j,n} - \delta_{j,-n}} E_{jj} && \text{for } \mathfrak{g} = B_n^{(1)}, \\
&= \sum_{j \in J} q^{\delta_{j,n-1} + \delta_{j,n} - \delta_{j,-n} - \delta_{j,-n+1}} E_{jj} && \text{for } \mathfrak{g} = D_n^{(1)}.
\end{aligned}$$

For the vector representation linear maps satisfying (3.5) are given as follows:

$$\begin{aligned}
\sigma^{(i)} : V^{(1)} &\longrightarrow V^{(1)}, \\
\sigma^{(1)}(v_{\pm 1}) &= sv_{\mp 1}, & \sigma^{(1)}(v_j) &= v_j \quad (j \in J, j \neq \pm 1), \\
\sigma^{(2)}(v_{\pm n}) &= v_{\mp n}, & \sigma^{(2)}(v_j) &= v_j \quad (j \in J, j \neq \pm n), \\
\sigma^{(3)}(v_j) &= (-)^{j-1} v_{-(n+1-j)}, & \sigma^{(3)}(v_{-j}) &= (-)^{j-1} sv_{n+1-j} \quad (j > 0).
\end{aligned}$$

Let  $\{v_j^* \mid j \in J\}$  be the canonical dual basis of  $\{v_j \mid j \in J\}$ . Then we have the following isomorphism of  $U_q(\mathfrak{g})$ -modules.

$$C_{\pm}^{(1)} : V_{z\xi^{\mp 1}}^{(1)} \xrightarrow{\sim} (V_z^{(1)})^{*a^{\pm 1}}; \quad v_j \mapsto q^{\pm \bar{j}} v_{-j}^*. \quad (3.10)$$

Here  $\bar{j}$  is defined by

$$\bar{j} = \begin{cases} j & (j > 0) \\ n & (j = 0) \\ j + N & (j < 0). \end{cases} \quad (3.11)$$

Let us recall  $R$ -matrices for  $V^{(1)} \otimes V^{(1)}$  obtained in [26]. Since the choice of the coproduct is different, we need a slight modification of the expressions given there.  $\overline{R}^{(1,1)}(z)$  is given by

$$\begin{aligned}
\overline{R}^{(1,1)}(z) &= \sum_{i \neq 0} E_{ii} \otimes E_{ii} + \frac{q(1-z)}{1-q^2z} \sum_{i \neq \pm j} E_{ii} \otimes E_{jj} \\
&+ \frac{1-q^2}{1-q^2z} \left( \sum_{i \prec j, i \neq -j} + z \sum_{i \succ j, i \neq -j} \right) E_{ij} \otimes E_{ji} \\
&+ \frac{1}{(1-q^2z)(1-\xi z)} \sum_{i,j} a_{ij}(z) E_{ij} \otimes E_{-i,-j}. \quad (3.12)
\end{aligned}$$

Here

$$a_{ij}(z) = \begin{cases} (q^2 - \xi z)(1-z) + \delta_{i0}(1-q)(q+z)(1-\xi z) & (i = j) \\ (1-q^2)(q^{\bar{j}-\bar{i}}(z-1) + \delta_{i,-j}(1-\xi z)) & (i \prec j) \\ (1-q^2)z(\xi q^{\bar{j}-\bar{i}}(z-1) + \delta_{i,-j}(1-\xi z)) & (i \succ j). \end{cases}$$

For the values of  $\alpha^{(1,1)}(z)$  and  $\beta^{(1,1)}(z)$  we can obtain

$$\alpha^{(1,1)}(z) = \frac{(1 - \xi^{-2}z)(1 - q^{-2}\xi^{-1}z)(1 - q^2\xi^{-1}z)(1 - z)}{(1 - q^2\xi^{-2}z)(1 - \xi^{-1}z)^2(1 - q^{-2}z)},$$

$$\beta^{(1,1)}(z) = q^{-1} \frac{(q^2z; \xi^2)_\infty (\xi z; \xi^2)_\infty^2 (q^{-2}\xi^2z; \xi^2)_\infty}{(z; \xi^2)_\infty (q^{-2}\xi z; \xi^2)_\infty (q^2\xi z; \xi^2)_\infty (\xi^2z; \xi^2)_\infty}.$$

**3.4 Spin representations and their  $R$ -matrices.** We shall define the spin representations. They are the representations associated with the tail vertices in the Dynkin diagram. There is only one spin representation in the case of  $B_n^{(1)}$ , and there are two in the case of  $D_n^{(1)}$ . Consider the 2 dimensional vector space  $V_{1/2}$  spanned by  $v_{1/2}$  and  $v_{-1/2}$ . We define operators  $X^+, X^-, T$  acting on this space by

$$X^+v_\eta = v_{\eta+1}, \quad X^-v_\eta = v_{\eta-1}, \quad Tv_\eta = q^\eta v_\eta.$$

Here  $\eta = \pm 1/2$ , otherwise  $v_\eta$  is to be understood as 0. Set  $V^{(sp)} = V_{1/2}^{\otimes n}$ . For a base vector  $v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_n}$  ( $\varepsilon_1, \dots, \varepsilon_n = \pm 1/2$ ) we use a shorthand notation  $v_\varepsilon$  ( $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ ). The weight of  $v_\varepsilon$  is given by  $\sum_{i=1}^n \varepsilon_i \epsilon_i$ . We put  $\varepsilon^+ = (1/2, \dots, 1/2)$ ,  $\varepsilon^- = (-1/2, \dots, -1/2)$ . For  $i = 1, \dots, n$  and  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$   $\varepsilon^{(i)}$  (resp.  $\varepsilon_{(i)}$ ) stands for  $(\varepsilon_1, \dots, \varepsilon_i + 1, \dots, \varepsilon_n)$  (resp.  $(\varepsilon_1, \dots, \varepsilon_i - 1, \dots, \varepsilon_n)$ ).  $\varepsilon^{(i)(j)}$ ,  $\varepsilon_{(j)}^{(i)}$ , etc, are defined similarly.

We define an action of  $U'_q(\mathfrak{g})$  on this space as follows  $(\pi^{(sp)}(f_i) = \pi^{(sp)}(e_i)^t)$ :

$$\begin{aligned} \pi^{(sp)}(e_0) &= X^- \otimes X^- \otimes 1 \otimes \cdots \otimes 1, \\ \pi^{(sp)}(t_0) &= T^{-1} \otimes T^{-1} \otimes 1 \otimes \cdots \otimes 1, \\ \pi^{(sp)}(e_i) &= 1 \otimes \cdots \otimes X^+ \otimes X^- \otimes \cdots \otimes 1 \quad (1 \leq i \leq n-1), \\ \pi^{(sp)}(t_i) &= 1 \otimes \cdots \otimes T \otimes T^{-1} \otimes \cdots \otimes 1 \quad (1 \leq i \leq n-1), \\ \pi^{(sp)}(e_n) &= 1 \otimes \cdots \otimes 1 \otimes X^+ \quad \text{for } \mathfrak{g} = B_n^{(1)}, \\ &= 1 \otimes \cdots \otimes 1 \otimes X^+ \otimes X^+ \quad \text{for } \mathfrak{g} = D_n^{(1)}, \\ \pi^{(sp)}(t_n) &= 1 \otimes \cdots \otimes 1 \otimes T \quad \text{for } \mathfrak{g} = B_n^{(1)}, \\ &= 1 \otimes \cdots \otimes 1 \otimes T \otimes T \quad \text{for } \mathfrak{g} = D_n^{(1)}. \end{aligned}$$

For this representation linear maps satisfying (3.5) are given as follows:

$$\begin{aligned} \sigma^{(i)} : V^{(sp)} &\longrightarrow V^{(sp)}, \\ \sigma^{(1)}(v_{(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n)}) &= v_{(-\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n)}, \\ \sigma^{(2)}(v_{(\varepsilon_1 \cdots \varepsilon_{n-1} \varepsilon_n)}) &= v_{(\varepsilon_1 \cdots \varepsilon_{n-1} - \varepsilon_n)}, \\ \sigma^{(3)}(v_{(\varepsilon_1 \cdots \varepsilon_n)}) &= v_{(-\varepsilon_n \cdots -\varepsilon_1)}. \end{aligned}$$

In the case of  $B_n^{(1)}$   $V^{(sp)}$  is irreducible and is also denoted by  $(\pi^{(n)}, V^{(n)})$ . In the case of  $D_n^{(1)}$  it decomposes into two components. We denote by  $(\pi^{(n)}, V^{(n)})$  (resp.  $(\pi^{(n-1)}, V^{(n-1)})$ ) the irreducible representation of  $U'_q(D_n^{(1)})$  whose space contains the vector  $v_{\varepsilon+}$  (resp.  $v_{\varepsilon(n)+}$ ). We have

$$\begin{aligned} \sigma^{(1)}, \sigma^{(2)} : V^{(n)} &\longrightarrow V^{(n-1)} & \sigma^{(1)}, \sigma^{(2)} : V^{(n-1)} &\longrightarrow V^{(n)} \\ \sigma^{(3)} : V^{(n)} &\longrightarrow V^{(n-i)} & \sigma^{(3)} : V^{(n-1)} &\longrightarrow V^{(n+i-1)} \end{aligned}$$

where  $i = 0$  if  $n$  is even and  $i = 1$  if  $n$  is odd.

Denoting by  $\{v_\varepsilon^*\}$  the canonical dual basis of  $\{v_\varepsilon\}$ , we have the following isomorphism of  $U_q(\mathfrak{g})$ -modules.

$$C_\pm^{(sp)} : V_{z\xi^{\mp 1}}^{(sp)} \xrightarrow{\sim} V_z^{(sp)*a^{\pm 1}}; \quad v_\varepsilon \mapsto a_\pm(\varepsilon)v_{-\varepsilon}^*, \quad (3.13)$$

where

$$\begin{aligned} a_\pm(\varepsilon) &= (-)^{\sum_{j=1}^n (n+1-j)\bar{\varepsilon}_j} q^\pm \sum_{j=1}^n (n+1/2-j)\bar{\varepsilon}_j & \text{for } \mathfrak{g} = B_n^{(1)}, \\ &= (-q)^\pm \sum_{j=1}^n (n-j)\bar{\varepsilon}_j & \text{for } \mathfrak{g} = D_n^{(1)}, \end{aligned}$$

and  $\bar{\varepsilon}_j = 0$  ( $\varepsilon_j = 1/2$ ),  $= 1$  ( $\varepsilon_j = -1/2$ ). For each irreducible component in the case of  $D_n^{(1)}$ ,  $C_\pm^{(sp)}$  alternates the indices of the spaces just like  $\sigma^{(3)}$ .

We present the explicit forms of the  $R$ -matrices for  $V^{(1)} \otimes V^{(n)}$  and  $V^{(n)} \otimes V^{(1)}$ , which we denote by  $\bar{R}^{(1,n)}(z)$  and  $\bar{R}^{(n,1)}(z)$ . We take  $v_{\varepsilon+}$  in  $V^{(n)}$  as a reference vector for the both cases. For the case of  $D_n^{(1)}$  we may also need the  $R$ -matrices for  $V^{(1)} \otimes V^{(n-1)}$  and  $V^{(n-1)} \otimes V^{(1)}$ . Those are given by the same formulae as  $\bar{R}^{(1,n)}(z)$  and  $\bar{R}^{(n,1)}(z)$  except that they are acting on different spaces  $V^{(1)} \otimes V^{(n-1)}$  and  $V^{(n-1)} \otimes V^{(1)}$ . As a reference vector for  $V^{(n-1)}$  in the case of  $D_n^{(1)}$  we take  $v_{\varepsilon(n)+}$ .

For  $i \in J$  we set  $\varepsilon(i) = \varepsilon^{(i)}(i > 0)$ ,  $= \varepsilon(i = 0)$ ,  $= \varepsilon_{(-i)}(i < 0)$ .  $\varepsilon(i, j)$  ( $i, j \in J$ ) is defined recursively. For  $i, j = 1, \dots, n$  ( $i < j$ ) define  $n_{ij}^+(\varepsilon) = \#\{k \mid i < k < j, \varepsilon_k = 1/2\}$ ,  $n_{i0}^+(\varepsilon) = \#\{k \mid i < k, \varepsilon_k = 1/2\}$  and put  $n_{ji}^+(\varepsilon) = -n_{ij}^+(\varepsilon) - 1$  for  $i, j = 0, \dots, n$ . The forms of the  $R$ -matrices are given below.

$$\begin{aligned} \bar{R}^{(1,n)}(z) &= \sum_{i,\varepsilon} a_{i\varepsilon}(z) E_{ii} \otimes E_{\varepsilon\varepsilon} + \sum_{i \neq j, \varepsilon} b_{ij\varepsilon} c_{ij}(z) (\xi^{1/2} sz)^{\theta(i \succ j)} E_{ij} \otimes E_{\varepsilon\varepsilon(i, -j)}, \\ \bar{R}^{(n,1)}(z) &= \sum_{i,\varepsilon} a_{i\varepsilon}(z) E_{\varepsilon\varepsilon} \otimes E_{ii} + \sum_{i \neq j, \varepsilon} \bar{b}_{ij\varepsilon} c_{ij}(z) (\xi^{1/2} sz)^{\theta(i \prec j)} E_{\varepsilon\varepsilon(i, -j)} \otimes E_{ij}, \end{aligned}$$

where

$$a_{i\varepsilon}(z) = \begin{cases} 1 & (i\varepsilon|_i| > 0), \\ q \frac{1 + q^{-1}\xi^{1/2}sz}{1 + q\xi^{1/2}sz} & (i\varepsilon|_i| < 0), \\ q^{1/2} \frac{1 + \xi^{1/2}sz}{1 + q\xi^{1/2}sz} & (i = 0), \end{cases}$$

$$\begin{aligned}
b_{ij\varepsilon} &= q^{-\eta(i)+\eta(j)+\delta_{i0}} (-q)^{n_{|i||j|}^+(\varepsilon)}, \\
\bar{b}_{ij\varepsilon} &= q^{\eta(i)-\eta(j)+(\delta_{i0}+\delta_{j0})/2} (-q)^{n_{|j||i|}^+(\varepsilon)}, \\
\eta(i) &= \begin{cases} 0 & (i \geq 0), \\ h^\vee/2 + i + 1 & (i < 0), \end{cases} \\
c_{ij}(z) &= \begin{cases} \frac{1-q^2}{1+q\xi^{1/2}sz} & (ij \neq 0), \\ \sqrt{[2]_n} \frac{1-q}{1+q\xi^{1/2}sz} & (ij = 0). \end{cases}
\end{aligned}$$

Here  $\theta(S) = 1$  if  $S$  is true,  $= 0$  otherwise. In the summation  $i, j$  run over  $J$  and  $\varepsilon$  runs over  $(\pm 1/2, \dots, \pm 1/2)$ .  $E_{\varepsilon\varepsilon'}$  stands for a matrix unit in  $\text{End}(V^{(n)})$ . The calculation is long but straightforward. We omit it.

For these  $R$ -matrices we have

$$\begin{aligned}
\alpha^{(1,n)}(z) &= \alpha^{(n,1)}(z) = \frac{(1+q\xi^{-1/2}sz)(1+q^{-1}\xi^{-3/2}sz)}{(1+q^{-1}\xi^{-1/2}sz)(1+q\xi^{-3/2}sz)}, \\
\beta^{(1,n)}(z) &= \beta^{(n,1)}(z) = q^{-1/2} \frac{(-q\xi^{1/2}sz; \xi^2)_\infty (-q^{-1}\xi^{3/2}sz; \xi^2)_\infty}{(-q^{-1}\xi^{1/2}sz; \xi^2)_\infty (-q\xi^{3/2}sz; \xi^2)_\infty}.
\end{aligned}$$

#### 4. Fusion construction

Throughout this section,  $(\pi, V)$  stands for the vector representation of  $U'_q(\mathfrak{g})$  defined in section 3, and  $P$  denotes the transposition on  $V \otimes V$ . We shall construct the fundamental representations  $(\pi^{(k)}, V^{(k)})$  of  $U'_q(\mathfrak{g})$  via the fusion construction, where  $2 \leq k \leq n-1$  for  $\mathfrak{g} = B_n^{(1)}$  and  $2 \leq k \leq n-2$  for  $\mathfrak{g} = D_n^{(1)}$ .

4.1 *Definition of  $V^{(k)}$ .* Since  $\bar{R}^{(1,1)}(z)$  has a pole at  $z = q^{-2}$ , we set

$$R(z) = (1 - q^2 z) \bar{R}^{(1,1)}(z). \quad (4.1)$$

Let  $R_{ii+1}(z)$  be an operator on  $V^{\otimes k}$  acting as  $R(z)$  on the  $i$ -th and  $(i+1)$ -th components and as the identity on the other components, and let

$$\begin{aligned}
S_{ij} &= R_{j-1j}(q^{-2}) R_{j-2j}(q^{-4}) \cdots R_{ij}(q^{2i-2j}) \quad (i < j), \\
T^{(k)} &= S_{1k} \cdots S_{13} S_{12}.
\end{aligned} \quad (4.2)$$

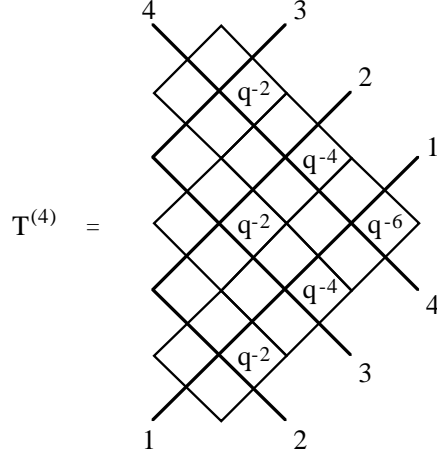
We define  $V^{(k)}$  by

$$V^{(k)} = V^{\otimes k} / \text{Ker } T^{(k)}.$$

A graphical representation of  $T^{(4)}$  is given in Figure 3.

We shall give a more concrete description of the space  $V^{(k)}$ . Set  $W = \text{Im } PR(q^2) \subset V^{\otimes 2}$ .





**Figure 3:** Graphical representation of  $T^{(4)}$

**Proposition 4.1.**  $W$  is generated by the following vectors.

$$\begin{aligned} &v_i \otimes v_i \quad (i \neq 0), \quad v_i \otimes v_j + qv_j \otimes v_i \quad (i \succ j, i \neq -j), \\ &v_{-i} \otimes v_i + q^2v_i \otimes v_{-i} - q(v_{i+1} \otimes v_{-i-1} + v_{-i-1} \otimes v_{i+1}) \quad (1 \leq i \leq n-1), \\ &v_{-n} \otimes v_n + q^2v_n \otimes v_{-n} - q(1+q)(v_0 \otimes v_0) \quad \text{for } \mathfrak{g} = B_n^{(1)}. \end{aligned}$$

The proof is given in the appendix.  $\text{Ker } T^{(k)}$  has the following concrete description.

**Proposition 4.2.**

$$\text{Ker } T^{(k)} = \sum_{j=0}^{k-2} V^{\otimes j} \otimes W \otimes V^{\otimes (k-2-j)},$$

Before proving this proposition we state a few properties of  $V^{(k)}$ . By virtue of these propositions it is easily seen that the vectors of the following form constitute a basis of  $V^{(k)}$ :

$$\begin{aligned} &v_{i_1} \otimes \cdots \otimes v_{i_a} \otimes v_0 \otimes \cdots \otimes v_0 \otimes v_{-j_b} \otimes \cdots \otimes v_{-j_1} \quad \text{for } \mathfrak{g} = B_n^{(1)}, \\ &v_{i_1} \otimes \cdots \otimes v_{i_a} \otimes v_{-n} \otimes v_n \otimes v_{-n} \otimes \cdots \otimes v_n \otimes v_{-j_b} \otimes \cdots \otimes v_{-j_1} \quad \text{for } \mathfrak{g} = D_n^{(1)}. \end{aligned} \quad (4.3)$$

Here  $1 \leq i_1 < \cdots < i_a \leq n, 1 \leq j_1 < \cdots < j_b \leq n$ . We shall call such a form the *normal order* form. Using this basis we easily get

$$\dim V^{(k)} = \text{rank } T^{(k)} = \sum_{j=0}^{[k/2]} \binom{N}{k-2j}.$$

Here  $N = \dim V$  and  $[i]$  denotes the largest integer that does not exceed  $i$ .

*Proof of Proposition 4.2.* By successive use of the Yang–Baxter equation (3.7) for  $R(z)$ , for any  $j$  ( $0 \leq j \leq k-2$ ), we can rewrite  $T^{(k)}$  into the following form:

$$T^{(k)} = T'_j R_{j+1j+2}(q^{-2}) \quad \text{with some operator } T'_j \text{ on } V^{\otimes k}.$$

Note that  $R(q^{-2})PR(q^2) = 0$ . So  $T^{(k)}$  kills a vector in  $V^{\otimes j} \otimes W \otimes V^{\otimes(k-2-j)}$  for any  $j$  ( $0 \leq j \leq k-2$ ). This shows  $\text{Ker } T^{(k)} \supset \sum_{j=0}^{k-2} V^{\otimes j} \otimes W \otimes V^{\otimes(k-2-j)}$ . To prove the other inclusion it suffices to show  $\text{rank } T^{(k)} \geq \sum_{j=0}^{[k/2]} \binom{N}{k-2j}$ . Put

$$T^{(k)} = \lim_{q \rightarrow 1} \frac{T^{(k)}}{(1 - q^2)^{k(k-1)/2}}.$$

To conclude it suffices to show

$$\text{rank } T^{(k)} \geq \sum_{j=0}^{[k/2]} \binom{N}{k-2j}, \quad (4.4)$$

which will be shown in the appendix.  $\square$

**4.2 Action of  $U_q(\mathfrak{g})$ .** We define the action of  $U_q(\mathfrak{g})$  on  $V_z^{(k)}$  below. Define

$$\begin{aligned} \pi_z^{(k)} : U_q(\mathfrak{g}) &\longrightarrow \text{End}(V^{\otimes k}) \\ \pi_z^{(k)}(x) &= (\pi_{(-q)^{1-k}z} \otimes \pi_{(-q)^{3-k}z} \otimes \cdots \otimes \pi_{(-q)^{k-1}z}) \Delta^{(k)}(x), \end{aligned}$$

where

$$\Delta^{(k)} = (\Delta \otimes \underbrace{\text{id} \otimes \cdots \otimes \text{id}}_{k-2}) \circ \Delta^{(k-1)}, \quad \Delta^{(2)} = \Delta.$$

In what follows we use running index notation.

$$\Delta^{(k)}(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(k)}.$$

**Proposition 4.3.**  $\pi_z^{(k)}(x)$  is well defined as an operator on  $V_z^{(k)}$ , that is,

$$\pi_z^{(k)}(x) \text{Ker } T^{(k)} \subset \text{Ker } T^{(k)}.$$

*Proof.* Put

$$\pi_z^{(k)'}(x) = (\pi_{(-q)^{1-k}z} \otimes \pi_{(-q)^{3-k}z} \otimes \cdots \otimes \pi_{(-q)^{k-1}z}) \sum x_{(k)} \otimes \cdots \otimes x_{(1)}.$$

Using the intertwining property of  $R(z)$  (3.6) we can show

$$T^{(k)} \pi_z^{(k)}(x) = \pi_z^{(k)'}(x) T^{(k)}.$$

This completes the proof.  $\square$

As a  $U_q(\bar{\mathfrak{g}})$ -module we have the following isomorphism:

$$V^{(k)} \cong V_{\bar{\Lambda}_k} \oplus V_{\bar{\Lambda}_{k-2}} \oplus \cdots \oplus (V_{\bar{\Lambda}_1} \text{ or } V_0), \quad (4.5)$$

where  $V_{\bar{\Lambda}_j}$  is the irreducible highest weight module with highest weight  $\bar{\Lambda}_j$  and  $V_0$  is the trivial module.

Next we consider the Dynkin diagram symmetries of the  $U'_q(\mathfrak{g})$ -module  $V^{(k)}$ :

$$\sigma^{(i)} : V^{(k)} \longrightarrow V^{(k)}.$$

We shall give the action of  $\sigma^{(i)}$  ( $i = 1, 2, 3$ ), by components, *i.e.*

$$\sigma^{(i)}(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sigma_1^{(i)}(v_{i_1}) \otimes \cdots \otimes \sigma_k^{(i)}(v_{i_k}),$$

where  $\sigma_m^{(i)}$ 's are given by the following rules.

$$\begin{aligned} \sigma_m^{(1)}(v_{\pm 1}) &= (-q)^{\pm(2m-k-1)} s v_{\mp 1}, \\ \sigma_m^{(1)}(v_j) &= v_j \quad (j \in J, j \neq \pm 1), \\ \sigma_m^{(2)}(v_{\pm n}) &= v_{\mp n}, \\ \sigma_m^{(2)}(v_j) &= v_j \quad (j \in J, j \neq \pm n), \\ \sigma_m^{(3)}(v_j) &= (-q)^{\frac{2m-k-1}{2}} (-)^{j-1} v_{-(n+1-j)}, \\ \sigma_m^{(3)}(v_{-j}) &= (-q)^{\frac{k+1-2m}{2}} (-)^{j-1} s v_{n+1-j} \quad (j > 0). \end{aligned}$$

Extending (3.10) we have an isomorphism of  $U_q(\mathfrak{g})$ -modules

$$C_{\pm}^{(k)} : V_{z\xi^{\mp 1}}^{(k)} \xrightarrow{\sim} (V_z^{(k)})^{*a^{\pm 1}}. \quad (4.6)$$

The proof is given in the appendix.

**4.3 Definition of  $\bar{R}^{(1,k)}(z)$  and  $\bar{R}^{(k,1)}(z)$ .** We define operators

$$\begin{aligned} R^{(1,k)}(z) &\in \text{End}(V \overset{0}{\otimes} V \overset{1}{\otimes} \cdots \overset{k}{\otimes} V), \\ R^{(k,1)}(z) &\in \text{End}(V \overset{1}{\otimes} \cdots \overset{k}{\otimes} V \overset{k+1}{\otimes} V) \end{aligned}$$

by

$$\begin{aligned} R^{(1,k)}(z) &= R_{0k}((-q)^{-k+1}z) \cdots R_{02}((-q)^{k-3}z) R_{01}((-q)^{k-1}z), \\ R^{(k,1)}(z) &= R_{1k+1}((-q)^{-k+1}z) \cdots R_{k-1k}((-q)^{k-3}z) R_{kk+1}((-q)^{k-1}z). \end{aligned}$$

**Proposition 4.4.**  $R^{(1,k)}(z)$  (resp.  $R^{(k,1)}(z)$ ) is well defined as an operator on  $V \otimes V^{(k)}$  (resp.  $V^{(k)} \otimes V$ ), that is,

$$\begin{aligned} R^{(1,k)}(z)(V \otimes \text{Ker } T^{(k)}) &\subset V \otimes \text{Ker } T^{(k)}, \\ R^{(k,1)}(z)(\text{Ker } T^{(k)} \otimes V) &\subset \text{Ker } T^{(k)} \otimes V. \end{aligned}$$

*Proof.* These can be proved from the following formulae:

$$\begin{aligned} (\text{id}_V \otimes T^{(k)}) R^{(1,k)}(z) &= R_{01}((-q)^{k-1}z) \cdots R_{0k}((-q)^{-k+1}z) (\text{id}_V \otimes T^{(k)}), \\ (T^{(k)} \otimes \text{id}_V) R^{(k,1)}(z) &= R_{kk+1}((-q)^{k-1}z) \cdots R_{1k+1}((-q)^{-k+1}z) (T^{(k)} \otimes \text{id}_V), \end{aligned}$$

which are shown easily by successive use of the Yang–Baxter equation (3.7).  $\square$

We shall show that  $R^{(1,k)}$  and  $R^{(k,1)}$  have the intertwining property in the sense of (3.6).

**Proposition 4.5.**

$$\begin{aligned}
R^{(1,k)}(z_1/z_2)(\pi_{z_1} \otimes \pi_{z_2}^{(k)})\Delta(x) &= (\pi_{z_1} \otimes \pi_{z_2}^{(k)})\Delta'(x)R^{(1,k)}(z_1/z_2) \quad \text{on } V \otimes V^{(k)}, \\
R^{(k,1)}(z_1/z_2)(\pi_{z_1}^{(k)} \otimes \pi_{z_2})\Delta(x) &= (\pi_{z_1}^{(k)} \otimes \pi_{z_2})\Delta'(x)R^{(k,1)}(z_1/z_2) \quad \text{on } V^{(k)} \otimes V.
\end{aligned}$$

*Proof.* The upper formula can be shown in the following manner.

$$\begin{aligned}
R^{(1,k)}(z) &\sum \pi_{z_1}(x_{(1)}) \otimes \pi_{z_2}^{(k)}(x_{(2)}) \\
&= R_{0k}((-q)^{-k+1}z) \cdots R_{01}((-q)^{k-1}z) \\
&\quad \sum \pi_{z_1}(x_{(1)}) \otimes \pi_{(-q)^{1-k}z_2}(x_{(2)}) \otimes \cdots \otimes \pi_{(-q)^{k-1}z_2}(x_{(k+1)}) \\
&= R_{0k}((-q)^{-k+1}z) \cdots R_{02}((-q)^{k-3}z) \\
&\quad \sum \pi_{z_1}(x_{(2)}) \otimes \pi_{(-q)^{1-k}z_2}(x_{(1)}) \otimes \cdots \otimes \pi_{(-q)^{k-1}z_2}(x_{(k+1)}) R_{01}((-q)^{k-1}z) \\
&\quad \vdots \\
&= \sum \pi_{z_1}(x_{(k+1)}) \otimes \pi_{(-q)^{1-k}z_2}(x_{(1)}) \otimes \cdots \otimes \pi_{(-q)^{k-1}z_2}(x_{(k)}) R^{(1,k)}(z) \\
&= \sum \pi_{z_1}(x_{(2)}) \otimes \pi_{z_2}^{(k)}(x_{(1)}) R^{(1,k)}(z).
\end{aligned}$$

The other formula can be shown similarly.  $\square$

As a reference vector for  $V^{(k)}$  we take  $v_{[12\dots k]}$ , where  $v_{[i_1\dots i_k]} = v_{i_1} \otimes \cdots \otimes v_{i_k}$ . We want to normalise  $R^{(1,k)}(z)$  (resp.  $R^{(k,1)}(z)$ ) so that

$$\begin{aligned}
R^{(1,k)}(z)v_1 \otimes v_{[12\dots k]} &= v_1 \otimes v_{[12\dots k]} \\
(\text{resp. } R^{(k,1)}(z)v_{[12\dots k]} \otimes v_1 &= v_{[12\dots k]} \otimes v_1).
\end{aligned}$$

**Proposition 4.6.**

$$\begin{aligned}
\overline{R}^{(1,k)}(z) &= \frac{1}{q^{k-1}(1 - (-q)^{k+1}z) \prod_{j=1}^{k-1} (1 - (-q)^{k-1-2j}z)} R^{(1,k)}(z), \\
\overline{R}^{(k,1)}(z) &= \frac{1}{q^{k-1}(1 - (-q)^{k+1}z) \prod_{j=1}^{k-1} (1 - (-q)^{k-1-2j}z)} R^{(k,1)}(z).
\end{aligned}$$

*Proof.*  $R^{(1,k)}(z)v_1 \otimes v_{[12\dots k]}$  produces many types of vectors in  $V^{\otimes k}$ . One such vector is obtained by applying each  $R_{0j}((-q)^{k+1-2j}z)$  diagonally. It is

$$\begin{aligned}
R^{(1,k)}(z)(v_1 \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_k) \\
= (1 - (-q)^{k+1}z) \cdot q(1 - (-q)^{k-3}z) \cdot q(1 - (-q)^{k-5}z) \\
\quad \cdots q(1 - (-q)^{-k+1}z)(v_1 \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_k).
\end{aligned}$$

All of the off-diagonal terms are zero in the quotient space  $V^{(k)}$  due to the definition of  $R^{(1,k)}(z)$  as a product  $R_{0k} \cdots R_{01}$ . For example,  $R_{02}(v_1 \otimes v_{[12\dots k]})$  produces two

terms, the diagonal one above and  $(v_2 \otimes v_{[11\dots k]})$ . The latter term is zero in  $V^{(k)}$ . The argument proceeds step by step. The other case can be shown similarly.  $\square$

4.4 *Image of the universal  $R$ .* Proceeding exactly as for the vector and spin representations, we find

$$\begin{aligned}\alpha^{(1,k)}(z) &= \alpha^{(k,1)}(z) \\ &= \frac{(1 - (-q)^{k-1}\xi^{-2}z)(1 - (-q)^{-k-1}\xi^{-1}z)(1 - (-q)^{k+1}\xi^{-1}z)(1 - (-q)^{-k+1}z)}{(1 - (-q)^{k+1}\xi^{-2}z)(1 - (-q)^{-k+1}\xi^{-1}z)(1 - (-q)^{k-1}\xi^{-1}z)(1 - (-q)^{-k-1}z)}, \\ \beta^{(1,k)}(z) &= \beta^{(k,1)}(z) \\ &= q^{-1} \frac{((-q)^{k+1}z; \xi^2)_\infty ((-q)^{-k+1}\xi z; \xi^2)_\infty ((-q)^{k-1}\xi z; \xi^2)_\infty ((-q)^{-k-1}\xi^2 z; \xi^2)_\infty}{((-q)^{k-1}z; \xi^2)_\infty ((-q)^{-k-1}\xi z; \xi^2)_\infty ((-q)^{k+1}\xi z; \xi^2)_\infty ((-q)^{-k+1}\xi^2 z; \xi^2)_\infty}.\end{aligned}$$

## 5. Vertex operators

5.1 *Definition of VOs.* Let  $V(\Lambda_i)$  be the irreducible highest weight module with highest weight  $\Lambda_i$ . Let  $|\Lambda_i\rangle$  be a highest weight vector of  $V(\Lambda_i)$ . As motivated in section 2 we consider level 1 modules only, *i.e.*  $i = 0, 1, n$  (and  $n - 1$  for  $\mathfrak{g} = D_n^{(1)}$ ). Let  $\lambda, \mu$  stand for level 1 weights. We define two types of VOs as intertwiners of the following  $U_q(\mathfrak{g})$ -modules.

$$\tilde{\Phi}_\lambda^{\mu V^{(k)}}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z^{(k)}, \quad (5.1a)$$

$$\tilde{\Phi}_\lambda^{V^{(k)}\mu}(z) : V(\lambda) \longrightarrow V_z^{(k)} \otimes V(\mu). \quad (5.1b)$$

Set  $\Delta_\lambda = (\lambda|\lambda + 2\rho)/2(h^\vee + 1)$ . We shift the powers of  $z$  by these ‘conformal factors’:

$$\Phi_\lambda^{\mu V^{(k)}}(z) = z^{\Delta_\mu - \Delta_\lambda} \tilde{\Phi}_\lambda^{\mu V^{(k)}}(z), \quad \Phi_\lambda^{V^{(k)}\mu}(z) = z^{\Delta_\mu - \Delta_\lambda} \tilde{\Phi}_\lambda^{V^{(k)}\mu}(z),$$

and call them type I VOs and type II VOs respectively. For a VO (5.1a) we define its ‘leading term’  $v_{lt}$  by

$$\tilde{\Phi}_\lambda^{\mu V^{(k)}}(z)|\lambda\rangle = |\mu\rangle \otimes v_{lt} + \dots$$

Here the terms in  $\dots$  should be of the form  $|u\rangle \otimes v$  with  $|u\rangle \in \bigoplus_{\xi \neq \mu} V(\mu)_\xi$ . Set

$$(V^{(k)})_\lambda^\mu = \{v \in V^{(k)} \mid \lambda \equiv \mu + \text{wt } v \pmod{\delta}, e_i^{\langle h_i, \mu \rangle + 1} v = 0 \text{ for all } i\},$$

The following criterion for the existence of VOs is known [20].

**Proposition 5.1.** *Mapping a VO to its leading term gives an isomorphism of vector spaces.*

$$\{VOs : V(\lambda) \rightarrow V(\mu) \otimes V_z^{(k)}\} \xrightarrow{\sim} (V^{(k)})_\lambda^\mu$$

The criterion for the existence of type II VOs is similar.

**5.2 Two point functions.** Here we collect useful propositions to calculate needed two point functions. Most of these come from appendix 1 of [3]. We consider the following three types of two point functions.

$$\langle \nu | \Phi_\mu^{\nu W_2}(z_2) \Phi_\lambda^{\mu V_1}(z_1) | \lambda \rangle \quad (5.2a)$$

$$\langle \nu | \Phi_\mu^{W_2 \nu}(z_2) \Phi_\lambda^{\mu V_1}(z_1) | \lambda \rangle \quad (5.2b)$$

$$\langle \nu | \Phi_\mu^{\nu W_2}(z_2) \Phi_\lambda^{V_1 \mu}(z_1) | \lambda \rangle \quad (5.2c)$$

Following [3], the space indexed by 1 (resp. 2) should always come in the first (resp. second) component of the tensor product. For instance, (5.2a) stands for the expectation value of the following composition of VOs.

$$V(\lambda) \xrightarrow{\Phi_\lambda^{\mu V}(z_1)} V(\mu) \otimes V_{z_1} \xrightarrow{\Phi_\mu^{\nu W}(z_1) \otimes 1} V(\nu) \otimes W_{z_2} \otimes V_{z_1} \xrightarrow{1 \otimes P} V(\nu) \otimes V_{z_1} \otimes W_{z_2}.$$

(5.2b) is similar, but in (5.2c) we don't need the last transposition. We call a two point function of type (5.2a), (5.2b), (5.2c) the type (I,I), (I,II), (II,I) two point function, respectively.

For finite dimensional representations  $\pi^V : U'_q(\mathfrak{g}) \rightarrow \text{End}(V)$ ,  $\pi^W : U'_q(\mathfrak{g}) \rightarrow \text{End}(W)$ , we put

$$R_+^{VW}(z) = (\pi^V \otimes \pi^W)(\mathcal{R}'(z)).$$

Then the two point functions are determined as solutions of the  $q$ -KZ equation (with appropriate analyticity properties):

**Proposition 5.2** [3,19]. *Let  $\Psi(z_1, z_2)$  be one of (5.2a – c). Then we have*

$$\Psi(pz_1, z_2) = A(z_1/z_2)\Psi(z_1, z_2), \quad \Psi(pz_1, pz_2) = (q^{-\phi} \otimes q^{-\phi})\Psi(z_1, z_2),$$

where  $p = q^{2(h^\vee + 1)}$ ,  $\phi = \bar{\lambda} + \bar{\nu} + 2\bar{\rho}$  and

$$A(z) = R_+^{VW}(pz)(q^{-\phi} \otimes 1) \quad \text{for (5.2a)}$$

$$= (q^{-\bar{\nu}} \otimes 1)R_+^{VW}(pq^{-1}z)(q^{-\phi + \bar{\nu}} \otimes 1) \quad \text{for (5.2b)}$$

$$= (q^{-\phi + \bar{\nu}} \otimes 1)R_+^{VW}(qz)(q^{-\bar{\nu}} \otimes 1) \quad \text{for (5.2c)}$$

For our calculation the following propositions are quite useful.

**Proposition 5.3** [3].

(i) Consider  $\Psi(z_1, z_2)$  in the case (5.2a). Then for any  $i$  we have

$$(\pi_{z_1}^V \otimes \pi_{z_2}^W)\Delta'(e_i)^{\langle h_i, \nu \rangle + 1}\Psi(z_1, z_2) = 0, \quad \text{wt } \Psi(z_1, z_2) = \bar{\lambda} - \bar{\nu}.$$

(ii) Let  $\Psi(z_1, z_2)$  be a solution for (5.2a). Then the following give solutions for the other cases:

$$(q^{-\bar{\nu}} \otimes 1)\Psi(q^{-1}z_1, z_2) \quad \text{for (5.2b)}$$

$$(q^{-\phi + \bar{\nu}} \otimes 1)\Psi(p^{-1}qz_1, z_2) \quad \text{for (5.2c)}$$

**Proposition 5.4.** Assume that for a  $V \otimes W$ -valued function  $w(z)$  we have

$$(\pi_{z_1}^V \otimes \pi_{z_2}^W) \Delta'(e_i)^{\langle h_i, \nu \rangle + 1} w(z_1/z_2) = 0 \quad \text{for all } i, \quad (5.3)$$

$$\overline{R}^{VW}(pz)(q^{-\phi} \otimes 1)w(z) = r(z)w(pz), \quad (5.4)$$

with some scalar function  $r(z)$ . Then setting  $\overline{w}(z) = P(q^{-\phi} \otimes 1)w(p^{-1}z^{-1})$ , we get

$$\begin{aligned} (\pi_{z_1}^W \otimes \pi_{z_2}^V) \Delta'(e_i)^{\langle h_i, \nu \rangle + 1} \overline{w}(z_1/z_2) &= 0 \quad \text{for all } i, \\ \overline{R}^{WV}(pz)(q^{-\phi} \otimes 1)\overline{w}(z) &= q^{-\langle \phi, \text{wt} \overline{w} \rangle} r(p^{-2}z^{-1})^{-1} \overline{w}(pz). \end{aligned}$$

*Proof.* From (5.3), for any  $i$  we have

$$\overline{R}^{WV}(z^{-1})P(\pi_{z_1}^V \otimes \pi_{z_2}^W) \Delta'(e_i)^{\langle h_i, \nu \rangle + 1} w(z_1/z_2) = 0.$$

The LHS is equal to

$$(\pi_{z_2}^W \otimes \pi_{z_1}^V) \Delta'(e_i)^{\langle h_i, \nu \rangle + 1} \overline{R}^{WV}(z^{-1})Pw(z).$$

Here using the first inversion relation and (5.4) we can get

$$\begin{aligned} \overline{R}^{WV}(z)Pw(z^{-1}) &= P\overline{R}^{VW}(z^{-1})^{-1}w(z^{-1}) \\ &= r(p^{-1}z^{-1})^{-1}P(q^{-\phi} \otimes 1)w(p^{-1}z^{-1}). \end{aligned}$$

This shows the first equality. The second can also be shown using the same formulae.

□

**5.3 Normalisation of VOs.** In this subsection we shall fix the normalisations of VOs. We start with recalling Dynkin diagram automorphisms (3.1). For level 1 highest weight modules there exist linear maps

$$\begin{aligned} \sigma^{(1)} : V(\Lambda_i) &\longrightarrow V(\Lambda_{1-i}) & (i = 0, 1), \\ \sigma^{(2)} : V(\Lambda_{n-i}) &\longrightarrow V(\Lambda_{n+i-1}) & (i = 0, 1), \\ \sigma^{(3)} : V(\Lambda_i) &\longrightarrow V(\Lambda_{n-i}) & (i = 0, 1, n-1, n), \end{aligned} \quad (5.5)$$

such that, for arbitrary  $x \in U'_q(\mathfrak{g})$ ,  $|u\rangle \in V(\lambda)$ ,  $\sigma^{(i)}|u\rangle \in V(\sigma^{(i)}(\lambda))$ ,  $\sigma^{(i)}$  maps the fixed highest weight vector  $|\lambda\rangle$  to the highest weight vector  $|\sigma^{(i)}(\lambda)\rangle$  and preserves the algebra action:  $\sigma^{(i)}(x|u\rangle) = \sigma^{(i)}(x)\sigma^{(i)}(|u\rangle)$ . (For a level 1 weight  $\lambda$ ,  $\sigma(\lambda)$  stands for the corresponding weight by (5.5).) Note that we have again used the same notation as (3.1). From the Dynkin diagram symmetry, if there exists a unique intertwiner  $V(\lambda) \longrightarrow V(\mu) \otimes V_z^{(k)}$  (up to a scalar factor), then there is also such an intertwiner  $\sigma(V(\lambda)) \longrightarrow \sigma(V(\mu)) \otimes \sigma(V^{(k)})_z$ , which may be obtained by composition of the linear maps  $\sigma^{(i)}$  ( $i = 1, 2, 3$ ). The condition for uniqueness is

$$\dim(V^{(k)})_\lambda^\mu = 1,$$

and the VOs appearing herein have this property. If we choose the normalisation

$$\tilde{\Phi}_{\lambda}^{\mu V^{(k)}}(z)|\lambda\rangle = |\mu\rangle \otimes v_{lt} + \cdots \quad (v_{lt} : \text{leading term}),$$

then we can use uniqueness to fix the normalisation of  $\tilde{\Phi}_{\sigma(\lambda)}^{\sigma(\mu)\sigma(V^{(k)})}(z)$  by

$$\tilde{\Phi}_{\sigma(\lambda)}^{\sigma(\mu)\sigma(V^{(k)})}(z)\sigma(|\lambda\rangle) = \sigma(|\mu\rangle) \otimes \sigma(v_{lt}) + \cdots.$$

We shall use this extensively to minimise calculations.

For the vector representation  $V^{(1)}$  we fix the normalisations as follows:

$$\begin{aligned} \tilde{\Phi}_{\Lambda_1}^{\Lambda_0 V^{(1)}}(z)|\Lambda_1\rangle &= |\Lambda_0\rangle \otimes v_1 + \cdots & \text{for } \mathfrak{g} = B_n^{(1)}, D_n^{(1)}, \\ \tilde{\Phi}_{\Lambda_n}^{\Lambda_n V^{(1)}}(z)|\Lambda_n\rangle &= |\Lambda_n\rangle \otimes \alpha^{-1}v_0 + \cdots & \text{for } \mathfrak{g} = B_n^{(1)}, \end{aligned}$$

where

$$\alpha = \sqrt{[2]_n}. \quad (5.6)$$

The VOs  $\tilde{\Phi}_{\Lambda_0}^{\Lambda_1 V^{(1)}}(\mathfrak{g} = B_n^{(1)}, D_n^{(1)})$  and  $\tilde{\Phi}_{\Lambda_{n-1}}^{\Lambda_n V^{(1)}}, \tilde{\Phi}_{\Lambda_n}^{\Lambda_{n-1} V^{(1)}}(\mathfrak{g} = D_n^{(1)})$  are normalised using Dynkin diagram automorphisms.

For the spin representations we normalise by

$$\begin{aligned} \tilde{\Phi}_{\Lambda_n}^{\Lambda_0 V^{(n)}}(z)|\Lambda_n\rangle &= |\Lambda_0\rangle \otimes v_{\varepsilon+} + \cdots & \text{for } \mathfrak{g} = B_n^{(1)}, D_n^{(1)}, \\ \tilde{\Phi}_{\Lambda_0}^{\Lambda_n V^{(n)}}(z)|\Lambda_0\rangle &= |\Lambda_n\rangle \otimes v_{\varepsilon-} + \cdots & \text{for } \mathfrak{g} = B_n^{(1)}. \end{aligned}$$

The following VOs are normalised using Dynkin diagram automorphisms.

$$\begin{aligned} B_n^{(1)} : & \tilde{\Phi}_{\Lambda_n}^{\Lambda_1 V^{(n)}}, \tilde{\Phi}_{\Lambda_1}^{\Lambda_n V^{(n)}}, \\ D_n^{(1)} : & \tilde{\Phi}_{\Lambda_n}^{\Lambda_1 V^{(n-1)}}, \tilde{\Phi}_{\Lambda_{n-1}}^{\Lambda_0 V^{(n-1)}}, \tilde{\Phi}_{\Lambda_{n-1}}^{\Lambda_1 V^{(n)}}, \\ & \tilde{\Phi}_{\Lambda_0}^{\Lambda_n V^{(n-i)}}, \tilde{\Phi}_{\Lambda_0}^{\Lambda_{n-1} V^{(n+i-1)}}, \tilde{\Phi}_{\Lambda_1}^{\Lambda_n V^{(n+i-1)}}, \tilde{\Phi}_{\Lambda_1}^{\Lambda_{n-1} V^{(n-i)}}. \end{aligned}$$

where  $i = 0$  if  $n$  is even and  $i = 1$  if  $n$  is odd.

For the fusion representations we normalise as follows:

(i)  $B_n^{(1)}$  :

for  $k$ : even

$$\begin{aligned} \tilde{\Phi}_{\Lambda_0}^{\Lambda_0 V^{(k)}}(z)|\Lambda_0\rangle &= |\Lambda_0\rangle \otimes \sum_{1 \leq i_1 < \cdots < i_m \leq n} a_{[i_1 \cdots i_m \cdots -i_m \cdots -i_1]} v_{[i_1 \cdots i_m \cdots -i_m \cdots -i_1]} + \cdots, \\ a_{[i_1 \cdots i_m \cdots -i_m \cdots -i_1]} &= q^{i_1 + \cdots + i_m - (n+1)m} \frac{(1 - q^k)(1 - q^{k-2}) \cdots (1 - q^{k-2m+2})}{(1 + q)^m}, \end{aligned}$$



for  $k$ : odd

$$\begin{aligned}\widetilde{\Phi}_{\Lambda_1}^{\Lambda_0 V^{(k)}}(z)|\Lambda_1\rangle &= |\Lambda_0\rangle \otimes \sum_{1 < i_1 < \dots < i_m \leq n} a_{[1i_1 \dots i_m \dots -i_m \dots -i_1]} v_{[1i_1 \dots i_m \dots -i_m \dots -i_1]} + \dots, \\ a_{[1i_1 \dots i_m \dots -i_m \dots -i_1]} &= q^{i_1 + \dots + i_m - (n+1)m} \frac{(1 - q^{k-1})(1 - q^{k-3}) \dots (1 - q^{k-2m+1})}{(1 + q)^m},\end{aligned}$$

for  $k$ : all

$$\widetilde{\Phi}_{\Lambda_n}^{\Lambda_n V^{(k)}}(z)|\Lambda_n\rangle = |\Lambda_n\rangle \otimes \alpha^{-k} v_{[0 \dots 0]} + \dots$$

The symbol  $\dots$  between  $i_m$  and  $-i_m$  stands for either a sequence of pairs  $0, 0$ , or it may be empty, as in the ‘normal order’ of generating vectors defined in (4.3).

(ii)  $D_n^{(1)}$  :

for  $k$ : even

$$\begin{aligned}\widetilde{\Phi}_{\Lambda_0}^{\Lambda_0 V^{(k)}}(z)|\Lambda_0\rangle &= |\Lambda_0\rangle \otimes \sum_{1 \leq i_1 < \dots < i_m \leq n} a_{[i_1 \dots i_m \dots -i_m \dots -i_1]} v_{[i_1 \dots i_m \dots -i_m \dots -i_1]} + \dots, \\ a_{[i_1 \dots i_m \dots -i_m \dots -i_1]} &= q^{i_1 + \dots + i_m - nm} (1 - q^k)(1 - q^{k-2}) \dots (1 - q^{k-2m+2}),\end{aligned}$$

for  $k$ : odd

$$\begin{aligned}\widetilde{\Phi}_{\Lambda_1}^{\Lambda_0 V^{(k)}}(z)|\Lambda_1\rangle &= |\Lambda_0\rangle \otimes \sum_{1 < i_1 < \dots < i_m \leq n} a_{[1i_1 \dots i_m \dots -i_m \dots -i_1]} v_{[1i_1 \dots i_m \dots -i_m \dots -i_1]} + \dots, \\ a_{[1i_1 \dots i_m \dots -i_m \dots -i_1]} &= q^{i_1 + \dots + i_m - nm} (1 - q^{k-1})(1 - q^{k-3}) \dots (1 - q^{k-2m+1}).\end{aligned}$$

The symbol  $\dots$  between  $i_m$  and  $-i_m$  stands for either a sequence of pairs  $-n, n$ , or it may be empty, as in the ‘normal order’ of generating vectors. The following VOs are normalised using Dynkin diagram automorphisms.

$$\begin{aligned}B_n^{(1)} : & \widetilde{\Phi}_{\Lambda_1}^{\Lambda_1 V^{(2k)}}, \widetilde{\Phi}_{\Lambda_0}^{\Lambda_1 V^{(2k-1)}}, \\ D_n^{(1)} : & \widetilde{\Phi}_{\Lambda_1}^{\Lambda_1 V^{(2k)}}, \widetilde{\Phi}_{\Lambda_0}^{\Lambda_1 V^{(2k-1)}}, \widetilde{\Phi}_{\Lambda_n}^{\Lambda_n V^{(2k)}}, \widetilde{\Phi}_{\Lambda_{n-1}}^{\Lambda_n V^{(2k-1)}}, \widetilde{\Phi}_{\Lambda_{n-1}}^{\Lambda_{n-1} V^{(2k)}}, \widetilde{\Phi}_{\Lambda_n}^{\Lambda_{n-1} V^{(2k-1)}}.\end{aligned}$$

The type II VOs are normalised in the same manner.

## 6. Calculation of two point functions

In this section we calculate all the two point functions of the form (5.2) required for computation of the excitation spectrum. The simplification that one of the VOs is always for the vector representation is due to the fact that our underlying physical model is based on  $\overline{R}^{(1,1)}(z)$ .

**6.1 Dynkin diagram symmetry of two point functions.** Let  $\lambda, \mu$  be level 1 dominant integral weights, let  $v$  be a weight vector in the fundamental representation  $V^{(k)}$ . We define a non negative integer  $m(\lambda, \mu; v)$  as the minimal value of  $m_0$  satisfying

$$\begin{aligned} \lambda - \mu + \sum_{j=0}^n m_j \alpha_j &\equiv \text{wt } v \pmod{\mathbf{Z}\delta}, \\ m_j &\geq 0 \quad (j = 0, 1, \dots, n). \end{aligned}$$

Suppose we have a two point function of the following form:

$$\langle \Phi_{\mu}^{\nu(V^{(k')})_2}(z_2) \Phi_{\lambda}^{\mu(V^{(k)})_1}(z_1) \rangle = z_1^{\Delta_{\mu} - \Delta_{\lambda}} z_2^{\Delta_{\nu} - \Delta_{\mu}} \sum_i a_i(z_1/z_2) v_i \otimes v'_i.$$

Let  $\sigma$  be a Dynkin diagram automorphism. Since the VOs are normalised using Dynkin diagram automorphisms, we have

$$\begin{aligned} \langle \Phi_{\sigma(\mu)}^{\sigma(\nu)\sigma(V^{(k')})_2}(z_2) \Phi_{\sigma(\lambda)}^{\sigma(\mu)\sigma(V^{(k)})_1}(z_1) \rangle &= z_1^{\Delta_{\sigma(\mu)} - \Delta_{\sigma(\lambda)}} z_2^{\Delta_{\sigma(\nu)} - \Delta_{\sigma(\mu)}} \\ &\times \sum_i a_i(z_1/z_2) (z_1/z_2)^{m_i} \sigma(v_i) \otimes \sigma(v'_i), \end{aligned}$$

where  $m_i = m(\sigma(\lambda), \sigma(\mu); \sigma(v_i)) - m(\lambda, \mu; v_i)$ . This symmetry reduces the number of cases of two point functions to calculate. In the following subsections, we first list all of the cases which are needed, and then give the explicit formulae, omitting those which are obtained using the Dynkin diagram symmetry.

Let us explain how to minimise the calculation of two point functions. We are to calculate the following four kinds.

- (i)  $\langle \Phi_{\mu}^{\nu(V^{(k)})_2}(z_2) \Phi_{\lambda}^{\mu(V^{(1)})_1}(z_1) \rangle,$
- (ii)  $\langle \Phi_{\mu'}^{\nu(V^{(1)})_2}(z_2) \Phi_{\lambda}^{\mu'(V^{(k)})_1}(z_1) \rangle,$
- (iii)  $\langle \Phi_{\mu}^{(V^{(k)})_2\nu}(z_2) \Phi_{\lambda}^{\mu(V^{(1)})_1}(z_1) \rangle,$
- (iv)  $\langle \Phi_{\mu'}^{\nu(V^{(1)})_2}(z_2) \Phi_{\lambda}^{\mu'(V^{(k)})_1}(z_1) \rangle,$

for  $k = 1, \dots, n$ . We start with (i). Using proposition 5.3 (i), we can obtain the two point function up to scalar function of  $z_1, z_2$ . Apply proposition 5.2 (a) to determine the scalar part. Calculating (ii) is similar, but proposition 5.4 helps us determine the scalar function. For (iii) and (iv) we use proposition 5.3 (ii).

6.2 *Spin representations.* The following combinations of weights occur:

$$(\lambda, \mu, \mu', \nu) = (\Lambda_n, \Lambda_n, \Lambda_1, \Lambda_0) \quad k = n \quad \text{for } \mathfrak{g} = B_n^{(1)}, \quad (6.1a)$$

$$= (\Lambda_n, \Lambda_n, \Lambda_0, \Lambda_1) \quad n \quad \text{for } \mathfrak{g} = B_n^{(1)}, \quad (6.1b)$$

$$= (\Lambda_0, \Lambda_1, \Lambda_n, \Lambda_n) \quad n \quad \text{for } \mathfrak{g} = B_n^{(1)}, \quad (6.1c)$$

$$= (\Lambda_1, \Lambda_0, \Lambda_n, \Lambda_n) \quad n \quad \text{for } \mathfrak{g} = B_n^{(1)}, \quad (6.1d)$$

$$= (\Lambda_{n-1}, \Lambda_n, \Lambda_1, \Lambda_0) \quad n \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.1e)$$

$$= (\Lambda_n, \Lambda_{n-1}, \Lambda_1, \Lambda_0) \quad n-1 \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.1f)$$

$$= (\Lambda_{n-1}, \Lambda_n, \Lambda_0, \Lambda_1) \quad n-1 \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.1g)$$

$$= (\Lambda_n, \Lambda_{n-1}, \Lambda_0, \Lambda_1) \quad n \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.1h)$$

$$= (\Lambda_0, \Lambda_1, \Lambda_n, \Lambda_{n-1}) \quad n-i \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.1i)$$

$$= (\Lambda_1, \Lambda_0, \Lambda_n, \Lambda_{n-1}) \quad n+i-1 \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.1j)$$

$$= (\Lambda_0, \Lambda_1, \Lambda_{n-1}, \Lambda_n) \quad n+i-1 \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.1k)$$

$$= (\Lambda_1, \Lambda_0, \Lambda_{n-1}, \Lambda_n) \quad n-i \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.1l)$$

where  $i = 0$  if  $n$  is even and  $i = 1$  if  $n$  is odd. Due to the Dynkin diagram symmetry it suffices to give the formulae for (6.1a), (6.1c) and (6.1e).

$$(i) \quad \langle \Phi_\mu^{\nu(V^{(k)})_2}(z_2) \Phi_\lambda^{\mu(V^{(1)})_1}(z_1) \rangle = z_1^{\Delta_\mu - \Delta_\lambda} z_2^{\Delta_\nu - \Delta_\mu} \frac{(-sq\xi^{5/2} z_1/z_2; \xi^2)_\infty}{(-sq\xi^{3/2} z_1/z_2; \xi^2)_\infty} w(z_1/z_2),$$

where  $w(z)$  reads as follows:

$$w(z) = \alpha^{-1} v_0 \otimes v_{\varepsilon^+} - q^{1/2} \sum_{j=1}^n (-q)^{n-j} v_j \otimes v_{\varepsilon^+_{(j)}} \quad \text{for (6.1a),}$$

$$= \alpha^{-1} sq^n v_0 \otimes v_{\varepsilon^-} + s \sum_{j=1}^n q^{j-1} v_{-j} \otimes v_{\varepsilon^-(j)} \quad \text{for (6.1c),}$$

$$= v_{-n} \otimes v_{\varepsilon^+} + \sum_{j=1}^{n-1} (-q)^{n-j} v_j \otimes v_{\varepsilon^+_{(j)(n)}} \quad \text{for (6.1e).}$$

$$(ii) \quad \langle \Phi_{\mu'}^{\nu(V^{(1)})_2}(z_2) \Phi_\lambda^{\mu'(V^{(k)})_1}(z_1) \rangle = z_1^{\Delta_{\mu'} - \Delta_\lambda} z_2^{\Delta_\nu - \Delta_{\mu'}} \frac{(-sq\xi^{5/2} z_1/z_2; \xi^2)_\infty}{(-sq\xi^{3/2} z_1/z_2; \xi^2)_\infty} w(z_1/z_2),$$

where  $w(z)$  reads as follows:

$$w(z) = \alpha^{-1} (-q)^n v_{\varepsilon^+} \otimes v_0 + \sum_{j=1}^n (-q)^{j-1} v_{\varepsilon^+_{(j)}} \otimes v_j \quad \text{for (6.1a),}$$

$$= \alpha^{-1} v_{\varepsilon^-} \otimes v_0 + \sum_{j=1}^n q^{n-j+1/2} v_{\varepsilon^-(j)} \otimes v_{-j} \quad \text{for (6.1c),}$$

$$= (-q)^{n-1} v_{\varepsilon^+} \otimes v_{-n} + \sum_{j=1}^{n-1} (-q)^{j-1} v_{\varepsilon^+_{(j)(n)}} \otimes v_j \quad \text{for (6.1e).}$$

$$(iii) \quad \langle \Phi_{\mu}^{(V^{(k)})2\nu}(z_2) \Phi_{\lambda}^{\mu(V^{(1)})1}(z_1) \rangle = z_1^{\Delta_{\mu}-\Delta_{\lambda}} z_2^{\Delta_{\nu}-\Delta_{\mu}} \frac{(-s\xi^{5/2}z_1/z_2; \xi^2)_{\infty}}{(-s\xi^{3/2}z_1/z_2; \xi^2)_{\infty}} w(z_1/z_2),$$

where  $w(z)$  reads as follows:

$$w(z) = \alpha^{-1} v_0 \otimes v_{\varepsilon+} - q^{1/2} \sum_{j=1}^n (-q)^{n-j} v_j \otimes v_{\varepsilon+(j)}^+ \quad \text{for (6.1a),}$$

$$= \alpha^{-1} s q^{n-1/2} v_0 \otimes v_{\varepsilon-} + s \sum_{j=1}^n q^{j-1} v_{-j} \otimes v_{\varepsilon-(j)} \quad \text{for (6.1c),}$$

$$= v_{-n} \otimes v_{\varepsilon+} + \sum_{j=1}^{n-1} (-q)^{n-j} v_j \otimes v_{\varepsilon+(j)(n)}^+ \quad \text{for (6.1e).}$$

$$(iv) \quad \langle \Phi_{\mu'}^{\nu(V^{(1)})2}(z_2) \Phi_{\lambda}^{(V^{(k)})1\mu'}(z_1) \rangle = z_1^{\Delta_{\mu'}-\Delta_{\lambda}} z_2^{\Delta_{\nu}-\Delta_{\mu'}} \frac{(-s\xi^{1/2}z_1/z_2; \xi^2)_{\infty}}{(-s\xi^{-1/2}z_1/z_2; \xi^2)_{\infty}} w(z_1/z_2),$$

where  $w(z)$  reads as follows:

$$w(z) = \alpha^{-1} q^{1/2} (-q)^{-n} v_{\varepsilon+} \otimes v_0 + \sum_{j=1}^n (-q)^{1-j} v_{\varepsilon+(j)}^+ \otimes v_j \quad \text{for (6.1a),}$$

$$= \alpha^{-1} v_{\varepsilon-} \otimes v_0 + \sum_{j=1}^n q^{j-n-1/2} v_{\varepsilon-(j)} \otimes v_{-j} \quad \text{for (6.1c),}$$

$$= (-q)^{1-n} v_{\varepsilon+} \otimes v_{-n} + \sum_{j=1}^{n-1} (-q)^{1-j} v_{\varepsilon+(j)(n)}^+ \otimes v_j \quad \text{for (6.1e).}$$

**6.3 Vector and fusion representations.** Note that  $1 \leq k \leq n-1$  for  $B_n^{(1)}$  ( $n-2$  for  $D_n^{(1)}$ ) in this subsection. The following combinations of weights occur:

$$(\lambda, \mu, \mu', \nu) = (\Lambda_1, \Lambda_0, \Lambda_1, \Lambda_0) \quad k : \text{even} \quad \text{for } \mathfrak{g} = B_n^{(1)}, D_n^{(1)}, \quad (6.2a)$$

$$= (\Lambda_0, \Lambda_1, \Lambda_1, \Lambda_0) \quad \text{odd} \quad \text{for } \mathfrak{g} = B_n^{(1)}, D_n^{(1)}, \quad (6.2b)$$

$$= (\Lambda_1, \Lambda_0, \Lambda_0, \Lambda_1) \quad \text{odd} \quad \text{for } \mathfrak{g} = B_n^{(1)}, D_n^{(1)}, \quad (6.2c)$$

$$= (\Lambda_0, \Lambda_1, \Lambda_0, \Lambda_1) \quad \text{even} \quad \text{for } \mathfrak{g} = B_n^{(1)}, D_n^{(1)}, \quad (6.2d)$$

$$= (\Lambda_n, \Lambda_n, \Lambda_n, \Lambda_n) \quad \text{all} \quad \text{for } \mathfrak{g} = B_n^{(1)}, \quad (6.2e)$$

$$= (\Lambda_{n-1}, \Lambda_n, \Lambda_{n-1}, \Lambda_n) \quad \text{even} \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.2f)$$

$$= (\Lambda_n, \Lambda_{n-1}, \Lambda_{n-1}, \Lambda_n) \quad \text{odd} \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.2g)$$

$$= (\Lambda_{n-1}, \Lambda_n, \Lambda_n, \Lambda_{n-1}) \quad \text{odd} \quad \text{for } \mathfrak{g} = D_n^{(1)}, \quad (6.2h)$$

$$= (\Lambda_n, \Lambda_{n-1}, \Lambda_n, \Lambda_{n-1}) \quad \text{even} \quad \text{for } \mathfrak{g} = D_n^{(1)}. \quad (6.2i)$$

Due to the Dynkin diagram symmetry it suffices to give the formulae for (6.2a),(6.2b) and (6.2e).

$$(i) \quad \langle \Phi_\mu^{\nu(V^{(k)})_2}(z_2) \Phi_\lambda^{\mu(V^{(1)})_1}(z_1) \rangle \\ = z_1^{\Delta_\mu - \Delta_\lambda} z_2^{\Delta_\nu - \Delta_\mu} \frac{((-q)^{k+1} \xi^2 z_1/z_2; \xi^2)_\infty ((-q)^{-k+1} \xi^3 z_1/z_2; \xi^2)_\infty}{((-q)^{k+1} \xi z_1/z_2; \xi^2)_\infty ((-q)^{-k+1} \xi^2 z_1/z_2; \xi^2)_\infty} w^{(i)}(z_1/z_2),$$

where  $w^{(i)}(z)$  reads as follows:

For  $B_n^{(1)}$ :

$$\begin{aligned} w^{(1)}(z) &= \sum c_{[i_1 \dots i_m]} v_1 \otimes v_{[i_1 \dots i_m \dots -i_m \dots -i_1]} \\ &\quad + \sum_{j \neq 1} c_{-j[i_1 \dots i_m]} v_j \otimes v_{[1 i_1 \dots i_m \dots -i_m \dots -j \dots -i_1]} \\ &\quad + \sum_{j \neq 1} c_{j[i_1 \dots i_m]} v_{-j} \otimes v_{[1 i_1 \dots j \dots i_m \dots -i_m \dots -i_1]} \\ &\quad + \sum c_{0[i_1 \dots i_m]} v_0 \otimes v_{[1 i_1 \dots i_m \dots -i_m \dots -i_1]} \quad \text{for (6.2a),} \\ w^{(2)}(z) &= \sum c'_{-j[i_1 \dots i_m]} v_j \otimes v_{[i_1 \dots i_m \dots -i_m \dots -j \dots -i_1]} \\ &\quad + \sum c'_{j[i_1 \dots i_m]} v_{-j} \otimes v_{[i_1 \dots j \dots i_m \dots -i_m \dots -i_1]} \\ &\quad + \sum c'_{0[i_1 \dots i_m]} v_0 \otimes v_{[i_1 \dots i_m \dots -i_m \dots -i_1]} \quad \text{for (6.2b),} \\ w^{(3)}(z) &= \alpha^{-(k+1)} (1 - (-q)^k) \\ &\quad \times \sum_{j=1}^n (q^{n-j} v_j \otimes v_{[0 \dots 0 -j]} + q^{n+j-2} z v_{-j} \otimes v_{[j 0 \dots 0]}) \\ &\quad + \alpha^{-(k+1)} (1 + q^{2n} z) v_0 \otimes v_{[0 \dots 0]} \quad \text{for (6.2e).} \end{aligned}$$

The symbol  $\dots$  between  $i_m$  and  $-i_m$  stands for either a sequence of pairs  $0, 0$ , or it may be empty. The coefficients in the foregoing formulae are given by

$$\begin{aligned} c_{[i_1 \dots i_m]} &= q^{i_1 + \dots + i_m - (n+1)m} \\ &\quad \times \frac{(1 - q^k)(1 - q^{k-2}) \dots (1 - q^{k-2m+2})}{(1 + q)^m} \begin{cases} (1 + q^{2n-k} z) & (i_1 \neq 1) \\ (1 - q^{2n} z) & (i_1 = 1), \end{cases} \\ \left. \begin{matrix} c_{-j[i_1 \dots i_m]} \\ c_{j[i_1 \dots i_m]} \end{matrix} \right\} &= (-q)^{l+1} q^{i_1 + \dots + i_m - (n+1)m} \quad (i_l < j < i_{l+1}) \\ &\quad \times \frac{(1 - q^k)(1 - q^{k-2}) \dots (1 - q^{k-2m})}{(1 + q)^{m+1}} (1 + q^{-k}) z \begin{cases} q^{n-j} \\ q^{j-n-1}, \end{cases} \\ c_{0[i_1 \dots i_m]} &= (-q)^{m+1} q^{i_1 + \dots + i_m - (n+1)m} \\ &\quad \times \frac{(1 - q^k)(1 - q^{k-2}) \dots (1 - q^{k-2m})}{(1 + q)^{m+1}} (1 + q^{-k}) z, \end{aligned}$$

$$\begin{aligned}
\left. \begin{matrix} c'_{-j[i_1 \dots i_m]} \\ c'_{j[i_1 \dots i_m]} \end{matrix} \right\} &= s(-q)^l q^{i_1 + \dots + i_m - (n+1)m} & (i_l < j < i_{l+1}) \\
&\times \frac{(1 - q^{k-1})(1 - q^{k-3}) \dots (1 - q^{k-2m+1})}{(1 + q)^m} \begin{cases} q^{2n-j} \\ q^{j-1}, \end{cases} \\
c'_{0[i_1 \dots i_m]} &= s(-q)^m q^{i_1 + \dots + i_m - (n+1)m} \\
&\times \frac{(1 - q^{k-1})(1 - q^{k-3}) \dots (1 - q^{k-2m+1})}{(1 + q)^m} q^n.
\end{aligned}$$

For  $D_n^{(1)}$ :

$$\begin{aligned}
w^{(1)}(z) &= \sum c_{[i_1 \dots i_m]} v_1 \otimes v_{[i_1 \dots i_m \dots -i_m \dots -i_1]} \\
&+ \sum_{j \neq 1} c_{-j[i_1 \dots i_m]} v_j \otimes v_{[1 i_1 \dots i_m \dots -i_m \dots -j \dots -i_1]} \\
&+ \sum_{j \neq 1} c_{j[i_1 \dots i_m]} v_{-j} \otimes v_{[1 i_1 \dots j \dots i_m \dots -i_m \dots -i_1]} \quad \text{for (6.2a),} \\
w^{(2)}(z) &= \sum c'_{-j[i_1 \dots i_m]} v_j \otimes v_{[i_1 \dots i_m \dots -i_m \dots -j \dots -i_1]} \\
&+ \sum c'_{j[i_1 \dots i_m]} v_{-j} \otimes v_{[i_1 \dots j \dots i_m \dots -i_m \dots -i_1]} \quad \text{for (6.2b).}
\end{aligned}$$

The symbol  $\dots$  between  $i_m$  and  $-i_m$  stands for either a sequence of pairs  $-n, n$ , or it may be empty. The coefficients in the foregoing formulae are given by

$$\begin{aligned}
c_{[i_1 \dots i_m]} &= q^{i_1 + \dots + i_m - nm} \\
&\times (1 - q^k)(1 - q^{k-2}) \dots (1 - q^{k-2m+2}) \begin{cases} (1 + q^{2n-k-1}z) & (i_1 \neq 1) \\ (1 - q^{2n-1}z) & (i_1 = 1), \end{cases} \\
\left. \begin{matrix} c_{-j[i_1 \dots i_m]} \\ c_{j[i_1 \dots i_m]} \end{matrix} \right\} &= (-q)^{l+1} q^{i_1 + \dots + i_m - nm} & (i_l < j < i_{l+1}) \\
&\times (1 - q^k)(1 - q^{k-2}) \dots (1 - q^{k-2m})(1 + q^{-k})z \begin{cases} q^{n-j} \\ q^{j-n}, \end{cases} \\
\left. \begin{matrix} c'_{-j[i_1 \dots i_m]} \\ c'_{j[i_1 \dots i_m]} \end{matrix} \right\} &= s(-q)^l q^{i_1 + \dots + i_m - nm} & (i_l < j < i_{l+1}) \\
&\times (1 - q^{k-1})(1 - q^{k-3}) \dots (1 - q^{k-2m+1}) \begin{cases} q^{2n-j-1} \\ q^{j-1}. \end{cases}
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \langle \Phi_{\mu'}^{\nu(V^{(1)})_2}(z_2) \Phi_{\lambda}^{\mu'(V^{(k)})_1}(z_1) \rangle \\
= z_1^{\Delta_{\mu'} - \Delta_{\lambda}} z_2^{\Delta_{\nu} - \Delta_{\mu'}} \frac{((-q)^{k+1} \xi^2 z_1 / z_2; \xi^2)_{\infty} ((-q)^{-k+1} \xi^3 z_1 / z_2; \xi^2)_{\infty}}{((-q)^{k+1} \xi z_1 / z_2; \xi^2)_{\infty} ((-q)^{-k+1} \xi^2 z_1 / z_2; \xi^2)_{\infty}} \bar{w}^{(i)}(z_1 / z_2),
\end{aligned}$$

where  $\bar{w}^{(i)}(z)$  reads as follows:

$$\begin{aligned}
\bar{w}^{(1)}(z) &= q^2 \xi^2 z P(q^{-2\bar{p} - \bar{\Lambda}_1} \otimes 1) w^{(1)}(q^{-2} \xi^{-2} z^{-1}) & \text{for (6.2a),} \\
\bar{w}^{(2)}(z) &= P(q^{-2\bar{p}} \otimes 1) w^{(2)}(z) & \text{for (6.2b),} \\
\bar{w}^{(3)}(z) &= q \xi z P(q^{-2\bar{p} - 2\bar{\Lambda}_n} \otimes 1) w^{(3)}(q^{-2} \xi^{-2} z^{-1}) & \text{for (6.2e).}
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad & \langle \Phi_{\mu}^{(V^{(k)})_{2\nu}}(z_2) \Phi_{\lambda}^{\mu(V^{(1)})_1}(z_1) \rangle \\
&= z_1^{\Delta_{\mu}-\Delta_{\lambda}} z_2^{\Delta_{\nu}-\Delta_{\mu}} \frac{(-(-q)^k \xi^2 z_1/z_2; \xi^2)_{\infty} (-(-q)^{-k} \xi^3 z_1/z_2; \xi^2)_{\infty}}{(-(-q)^k \xi z_1/z_2; \xi^2)_{\infty} (-(-q)^{-k} \xi^2 z_1/z_2; \xi^2)_{\infty}} w(z_1/z_2),
\end{aligned}$$

where  $w(z)$  reads as follows:

$$\begin{aligned}
w(z) &= w^{(1)}(q^{-1}z) && \text{for (6.2a),} \\
&= w^{(2)}(z) && \text{for (6.2b),} \\
&= (q^{-\bar{\Lambda}_n} \otimes 1) w^{(3)}(q^{-1}z) && \text{for (6.2e).}
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad & \langle \Phi_{\mu'}^{\nu(V^{(1)})_2}(z_2) \Phi_{\lambda}^{(V^{(k)})_{1\mu'}}(z_1) \rangle \\
&= z_1^{\Delta_{\mu'}-\Delta_{\lambda}} z_2^{\Delta_{\nu}-\Delta_{\mu'}} \frac{(-(-q)^k z_1/z_2; \xi^2)_{\infty} (-(-q)^{-k} \xi z_1/z_2; \xi^2)_{\infty}}{(-(-q)^k \xi^{-1} z_1/z_2; \xi^2)_{\infty} (-(-q)^{-k} z_1/z_2; \xi^2)_{\infty}} \bar{w}(z_1/z_2),
\end{aligned}$$

where  $\bar{w}(z)$  reads as follows:

$$\begin{aligned}
\bar{w}(z) &= (q^{-2\bar{\rho}-\bar{\Lambda}_1} \otimes 1) \bar{w}^{(1)}(q^{-1}\xi^{-2}z) && \text{for (6.2a),} \\
&= \xi^{-1} (q^{-2\bar{\rho}} \otimes 1) \bar{w}^{(2)}(z) && \text{for (6.2b),} \\
&= (q^{-2\bar{\rho}-\bar{\Lambda}_n} \otimes 1) \bar{w}^{(3)}(q^{-1}\xi^{-2}z) && \text{for (6.2e).}
\end{aligned}$$

## 7. Commutation relations

**7.1 VOs for dual representations.** Here we define and normalise the VOs of type I  $\Phi_{\lambda}^{\mu(V^{(1)})^{*a\pm 1}}(z)$  and of type II  $\Phi_{\lambda}^{(V^{(k)})^{*a-1}\mu}(z)$ . They are used for the mathematical formulation of the local structure, transfer matrix and creation operators in 7.3.

For the former case, we recall the isomorphism  $C_{\pm}^{(1)}$  (3.10), and define

$$\tilde{\Phi}_{\lambda}^{\mu(V^{(1)})^{*a\pm 1}}(z) = (\text{const})(\text{id} \otimes C_{\pm}^{(1)}) \cdot \tilde{\Phi}_{\lambda}^{\mu V^{(1)}}(z\xi^{\mp 1}).$$

The constant prefactor is so chosen that we have

$$\tilde{\Phi}_{\lambda}^{\mu(V^{(1)})^{*a\pm 1}}(z)|\lambda\rangle = |\mu\rangle \otimes \gamma v_j^* + \cdots,$$

where  $v_j$  is a base vector in  $V^{(1)}$  such that  $\text{wt } v_j = \mu - \lambda$ , and  $\gamma = 1$  except that when  $\lambda = \mu = \Lambda_n$  for  $B_n^{(1)}$ , then  $\gamma = q^{-1/2}\alpha^{-1}$  ( $\alpha$  is defined in (5.6)). This choice of  $\gamma$  makes (7.2) and (7.4) hold for any possible pair  $(\lambda, \mu)$ .

For the VOs  $\Phi_{\lambda}^{(V^{(k)})^{*a-1}\mu}(z)$  of type II, using (3.13) and (4.6), we simply define

$$\begin{aligned}
\Phi_{\lambda}^{(V^{(k)})^{*a-1}\mu}(z) &= s(C_-^{(k')} \otimes \text{id}) \cdot \Phi_{\lambda}^{(V^{(k')})\mu}(z\xi^{-1}) && (k : \text{spin and } \Lambda_0 \in \{\lambda, \mu\}), \\
&= (C_-^{(k')} \otimes \text{id}) \cdot \Phi_{\lambda}^{(V^{(k')})\mu}(z\xi^{-1}) && (\text{otherwise}).
\end{aligned}$$

Here, if  $V^{(k)}$  is for the vector or fusion representation, then we need  $k' = k$ , but if it is for the spin representation, then  $C_-^{(k')}$  should be understood as  $C_-^{(sp)}$  and  $k'$  of  $V^{(k')}$  should be chosen carefully. (The isomorphism is explained directly after (3.13).)  $s$  is defined in (3.9). This definition is made so that part (3) of proposition 7.1 below will hold for all possible choices of  $(\lambda, \mu, \mu', \nu)$ , without any extra prefactors.

**7.2 Commutation relations.** Using the results of section 6 and definitions of subsections 5.3 and 7.1, we can prove the following commutation relations for VOs.

**Proposition 7.1.** *For any possible combination of weights  $(\lambda, \mu)$  or  $(\lambda, \mu, \mu', \nu)$ , we have*

$$\begin{aligned} (1) \quad & \Phi_\mu^{\lambda(V^{(1)})_2}(z_2) \Phi_\lambda^{\mu(V^{(1)})_1}(z_1) = r(z_1/z_2) \bar{R}^{(1,1)}(z_1/z_2) \Phi_\mu^{\lambda(V^{(1)})_1}(z_1) \Phi_\lambda^{\mu(V^{(1)})_2}(z_2), \\ (2) \quad & \Phi_{\mu'}^{\nu(V^{(1)})_2}(z_2) \Phi_\lambda^{(V^{(k)})_1 \mu'}(z_1) = \tau^{(k)}(z_1/z_2) \Phi_\mu^{(V^{(k)})_1 \nu}(z_1) \Phi_\lambda^{\mu(V^{(1)})_2}(z_2), \\ (3) \quad & \Phi_{\mu'}^{\nu(V^{(1)})_2}(z_2) \Phi_\lambda^{(V^{(k)})_1^{*a-1} \mu'}(z_1) = \tau^{(k)}(z_1/z_2)^{-1} \Phi_\mu^{(V^{(k)})_1^{*a-1} \nu}(z_1) \Phi_\lambda^{\mu(V^{(1)})_2}(z_2). \end{aligned}$$

Here

$$\begin{aligned} r(z) &= z^{-1} \frac{(q^2 z; \xi^2)_\infty (\xi z; \xi^2)_\infty (q^2 \xi z^{-1}; \xi^2)_\infty (\xi^2 z^{-1}; \xi^2)_\infty}{(q^2 \xi z; \xi^2)_\infty (\xi^2 z; \xi^2)_\infty (q^2 z^{-1}; \xi^2)_\infty (\xi z^{-1}; \xi^2)_\infty}, \\ \tau^{(k)}(z) &= z^{-1} \frac{\Theta_{\xi^2}(-(-q)^k z) \Theta_{\xi^2}(-(-q)^{-k} \xi z)}{\Theta_{\xi^2}(-(-q)^k z^{-1}) \Theta_{\xi^2}(-(-q)^{-k} \xi z^{-1})}, \quad (\text{fusion}) \\ &= z^{-1/2} \frac{\Theta_{\xi^2}(-s \xi^{1/2} z)}{\Theta_{\xi^2}(-s \xi^{1/2} z^{-1})}. \quad (\text{spin}) \end{aligned}$$

We only need to show them at the level of vacuum expectation value (two point function). See proposition 6.1 of [1].

**7.3 Mathematical formulations.** In this subsection, we briefly review the symmetry approach [1,3], and collect formulae needed for the validity of this approach.  $V$  is again to be understood as  $V^{(1)}$ .

In section 2 we have defined the space of states to be

$$\begin{aligned} \mathcal{F} &= \bigoplus_{i,j} \mathcal{F}_{\Lambda_i \Lambda_j}, \\ \mathcal{F}_{\lambda \mu} &= V(\lambda) \otimes V(\mu)^{*a} \simeq \text{Hom}_{\mathbf{Q}(q)}(V(\mu), V(\lambda)), \end{aligned}$$

where  $i, j$  run over  $0, 1, n$  (also  $n-1$  for  $D_n^{(1)}$ ). The vacuum (ground state) vector  $|\text{vac}\rangle_\lambda \in \mathcal{F}_{\lambda \lambda}$  is defined as  $\text{id}_{V(\lambda)}$ . Following [3], we define the left action of  $U_q(\mathfrak{g})$  on  $\mathcal{F}_{\lambda \mu}$ . We can also define the right action on the same underlying space. This right module is denoted by  $\mathcal{F}_{\lambda \mu}^r$ . There is a natural pairing:

$$\langle f | g \rangle = \frac{\text{tr}_{V(\lambda)}(q^{-2\rho} f g)}{\text{tr}_{V(\lambda)}(q^{-2\rho})}, \quad f \in \mathcal{F}_{\lambda \mu}^r, \quad g \in \mathcal{F}_{\mu \lambda}. \quad (7.1)$$



Next we show that  $\tilde{\Phi}_\lambda^{\mu V}(z)$  gives an isomorphism from  $V(\lambda)$  to  $V(\mu) \otimes V_z$ . Define

$$p: V^{*a} \otimes V \longrightarrow \mathbf{Q}(q), \quad v_1^* \otimes v_2 \mapsto \langle v_1^*, v_2 \rangle.$$

Then the above statement is a consequence of the following formulae. (See proposition 4.1 of [3].)

$$p \left( \langle \tilde{\Phi}_\mu^{\lambda V_1^{*a}}(z) \tilde{\Phi}_\lambda^{\mu V_2}(z) \rangle \right) = g, \quad (7.2a)$$

$$\langle \tilde{\Phi}_\mu^{\lambda V_1}(z) \tilde{\Phi}_\lambda^{\mu V_2^{*a}}(z) \rangle = g \sum_{i \in J} v_i \otimes v_i^*. \quad (7.2b)$$

Here

$$g = \frac{(q^2 \xi; \xi^2)_\infty (\xi^2; \xi^2)_\infty}{(q^2; \xi^2)_\infty (\xi; \xi^2)_\infty}. \quad (7.3)$$

Iterating this  $\tilde{\Phi}_\lambda^{\mu V}(z)$  we see the local structure of our space of states.

We proceed to the mathematical formulation to the row transfer matrix. Recall the definitions of  $\lambda^{(k)}, \lambda^{(k)*}$  in subsection 2.5 and the discussion in subsection 1.2. The row transfer matrix

$$T(z) = T_{\lambda\mu}^{\lambda^{(1)}\mu^{(1)}}(z): \mathcal{F}_{\lambda\mu} \longrightarrow \mathcal{F}_{\lambda^{(1)}\mu^{(1)}}$$

is formulated as the composition of the following operators:

$$V(\lambda) \otimes V(\mu)^{*a} \longrightarrow V(\lambda^{(1)}) \otimes V_z \otimes V(\mu)^{*a} \longrightarrow V(\lambda^{(1)}) \otimes V(\mu^{(1)})^{*a}.$$

Here the first and the second maps are given by  $\tilde{\Phi}_\lambda^{\lambda^{(1)}V}(z) \otimes \text{id}$  and  $\text{id} \otimes \left( \tilde{\Phi}_{\mu^{(1)}}^{\mu V^{*a-1}}(z) \right)^t$ .

We can show

$$T_{\lambda\lambda}^{\lambda^{(1)}\lambda^{(1)}}(z) |\text{vac}\rangle_\lambda = g |\text{vac}\rangle_{\lambda^{(1)}},$$

from the formula (See 4.3 of [3].):

$$\bar{p} \left( \langle \tilde{\Phi}_\mu^{\lambda V_1}(z) \tilde{\Phi}_\lambda^{\mu V_2^{*a-1}}(z) \rangle \right) = g, \quad (7.4)$$

where  $\bar{p}$  is defined by

$$\bar{p}: V \otimes V^{*a-1} \longrightarrow \mathbf{Q}(q), \quad v_1 \otimes v_2^* \mapsto \langle v_1, v_2^* \rangle.$$

Finally we recall the formulation of creation and annihilation operators. Let  $I$  be an index of base vectors in  $V^{(k)}$ . Decompose the following type II VOs into components:

$$\Phi_\lambda^{V^{(k)}\lambda^{(k)}}(z) = \sum_I v_I \otimes \Phi_{\lambda,I}^{(k)}(z),$$

$$\Phi_\lambda^{(V^{(k)})^{*a-1}\lambda^{(k)*}}(z) = \sum_I v_I^* \otimes \Phi_{\lambda,I}^{(k)*}(z).$$

The creation operator  $\phi_{\lambda,I}^{(k)*}(z)$  is defined by

$$\phi_{\lambda,I}^{(k)*}(z) : \mathcal{F}_{\lambda\mu} \longrightarrow \mathcal{F}_{\lambda^{(k)*}\mu}, \quad f \mapsto \Phi_{\lambda,I}^{(k)*}(z) \circ f.$$

The annihilation operator  $\phi_{\lambda,I}^{(k)}(z)$  is defined by the adjoint of

$$\mathcal{F}_{\mu\lambda^{(k)}}^r \longrightarrow \mathcal{F}_{\mu\lambda}^r, \quad f \mapsto f \circ \Phi_{\lambda,I}^{(k)}(z),$$

with respect to the pairing (7.1). In both cases the quasi-momentum  $z$  is supposed to be on  $|z| = 1$ .

In conclusion, the commutation relations between the transfer matrix and the creation (annihilation) operators given in section 2 are direct consequences of parts (2) and (3) of proposition 7.1.

## Appendix A

*A.1 Proof of proposition 4.1.* We shall describe  $\text{Im } R(q^2)$  rather than  $W$  (see (3.12) and (4.1) for the definition of  $R(z)$ ). Since  $R$ -matrices preserve the weight, we can concentrate on each weight space in  $V \otimes V$ . For the weight spaces of non-zero weight  $2\eta_1\epsilon_l$  and  $\eta_1\epsilon_l + \eta_2\epsilon_m$  ( $\eta_1, \eta_2 = \pm 1, l, m = 1, \dots, n$ ), it is easy to see that they are generated by the vectors  $v_i \otimes v_i$  and  $qv_i \otimes v_j + q^{2\theta(j \prec i)}v_j \otimes v_i$  ( $i = \eta_1 l, j = \eta_2 m$ ).

The weight space of weight 0 is generated by the following vectors:

$$A_j = \sum_{i \in J} \bar{a}_{ij} v_i \otimes v_{-i} \quad (j \in J),$$

where  $\bar{a}_{ij} = a_{ij}(q^2)/(1 - q^2)$ , and is given explicitly by

$$\bar{a}_{ij} = \begin{cases} q^2(1 - \xi) & (i = j \neq 0) \\ q^2(1 - \xi) + q(1 - q^2\xi) & (i = j = 0) \\ q^{\bar{j}-\bar{i}}(q^2 - 1) + \delta_{i,-j}(1 - q^2\xi) & (i \prec j) \\ q^2(\xi q^{\bar{j}-\bar{i}}(q^2 - 1) + \delta_{i,-j}(1 - q^2\xi)) & (i \succ j). \end{cases}$$

For convenience we put  $u_j = v_j \otimes v_{-j}$  ( $j \in J$ ). In what follows in the proof we assume  $1 \leq i, j \leq n$ , and for  $\mathfrak{g} = D_n^{(1)}$  we ignore the term containing  $u_0$  and also  $A_0$ . Noting  $\xi = q^{N-2}$  and recalling the definition (3.11) we get

$$q^{-j}A_j - q^jA_{-j} = (1 - q^N)\{(q^2 - 1) \sum_{i < j} (q^{-i}u_i + q^i u_{-i}) + (q^{2-j} - q^j)(u_j + u_{-j})\}.$$

Setting

$$B_j = (q^2 - 1) \sum_{i < j} (q^{-i}u_i + q^i u_{-i}) + (q^{2-j} - q^j)(u_j + u_{-j}),$$

we obtain

$$B_{j+1} - B_j = (q^j - q^{-j})(u_j + q^2 u_{-j} - q(u_{j+1} + u_{-j-1})).$$

Therefore we have

$$u_j + q^2 u_{-j} - q(u_{j+1} + u_{-j-1}) \in \text{Im } R(q^2) \quad (j = 1, \dots, n-1).$$

Let  $U$  be the subspace of  $\text{Im } R(q^2)$  generated by these vectors. Using  $(u_j + q^2 u_{-j}) \equiv q(u_{j+1} + u_{-j-1}) \pmod{U}$  we can show

$$\left. \begin{aligned} q^{-2j} A_j &\equiv \begin{cases} q^{n-j}(1-q)(u_n + q^2 u_{-n} - q(1+q)u_0) & \text{for } B_n^{(1)} \\ 0 & \text{for } D_n^{(1)} \end{cases} \\ A_0 &\equiv -(1-q^{2n})(u_n + q^2 u_{-n} - q(1+q)u_0) \end{aligned} \right\} \pmod{U}.$$

This completes the proof.  $\square$

A.2 Proof of (4.4). We define  $R^{(k)}$  by

$$R^{(k)} = \lim_{q \rightarrow 1} \frac{R(q^{-2k})}{1 - q^2}.$$

For the definition of  $R(z)$  see (3.12) and (4.1). Then from (4.2)  $T^{(k)}$  is obtained by

$$\begin{aligned} T^{(k)} &= \widehat{S}_{1k} \cdots \widehat{S}_{13} \widehat{S}_{12}, \\ \widehat{S}_{ij} &= R_{j-1j}^{(1)} R_{j-2j}^{(2)} \cdots R_{ij}^{(j-i)}. \end{aligned}$$

Calculating explicitly we have

$$R^{(k)} = P - kI + \frac{2k}{h^\vee - 2k} \sum_{i,j \in J} E_{ij} \otimes E_{-i-j}. \quad (\text{A1})$$

Let  $l$  be an integer such that  $0 \leq l \leq k$ ,  $k-l$  : even. Let  $v_{[i_1 \dots i_k]}$  stand for  $v_{i_1} \otimes \cdots \otimes v_{i_k}$  in  $V^{\otimes k}$  as opposed to section 4. We define for  $i_1, \dots, i_l \in J$

$$w_{[i_1 \dots i_l]}^{(k)} = \sum_{i_{l+1}, \dots, i_k \in J}^{(1)} \sum_{\tau \in S_k}^{(2)} \text{sgn } \tau \cdot v_{[i_{\tau(1)} \dots i_{\tau(k)}]}.$$

Here  $S_k$  denotes the  $k$ -th symmetric group, and the two summations are restricted as

$$\begin{aligned} \sum^{(1)} &: i_{l+1} + i_{l+2} = 0, \ i_{l+3} + i_{l+4} = 0, \ \dots, \ i_{k-1} + i_k = 0, \\ \sum^{(2)} &: \tau^{-1}(l+1) < \tau^{-1}(l+3) < \cdots < \tau^{-1}(k-1), \\ &\quad \tau^{-1}(l+1) < \tau^{-1}(l+2), \ \tau^{-1}(l+3) < \tau^{-1}(l+4), \\ &\quad \dots \quad \tau^{-1}(k-1) < \tau^{-1}(k). \end{aligned}$$

It is easy to see that  $w_{[i_1 \dots i_l]}^{(k)}$  has the following properties:

$$w_{[i_{\tau(1)} \dots i_{\tau(l)}]}^{(k)} = \text{sgn } \tau \cdot w_{[i_1 \dots i_l]}^{(k)} \quad \text{for } \tau \in S_l, \quad (\text{A2a})$$

$$w_{[i_1 \dots i_l]}^{(k)} = \sum_{1 \leq m \leq l} (-)^{m-1} v_{i_m} \otimes w_{[i_1 \dots \widehat{i_m} \dots i_l]}^{(k-1)} + (-)^l \sum_{p \in J} v_p \otimes w_{[i_1 \dots i_{l-p}]}^{(k-1)}, \quad (\text{A2b})$$

where  $\widehat{\phantom{x}}$  means omission. Now we prepare

**Lemma A1.**

$$\begin{aligned} \widehat{S}_{1k+1}(w_{[i_1 \dots i_l]}^{(k)} \otimes v_j) \\ = (-)^k k! \frac{h^\vee - k - l}{h^\vee - 2k} w_{[i_1 \dots i_l j]}^{(k+1)} + k! \frac{k - l + 2}{h^\vee - 2k} \sum_{1 \leq m \leq l} (-)^{m-1} \delta_{i_m, -j} w_{[i_1 \dots \widehat{i_m} \dots i_l]}^{(k+1)}. \end{aligned}$$

*Proof.* First, from (A2b) and (A1) we have

$$\begin{aligned} R_{1k+1}^{(k)}(w_{[i_1 \dots i_l]}^{(k)} \otimes v_j) \\ = \sum_{1 \leq m \leq l} (-)^{m-1} \left( v_j \otimes w_{[i_1 \dots \widehat{i_m} \dots i_l]}^{(k-1)} \otimes v_{i_m} - k v_{i_m} \otimes w_{[i_1 \dots \widehat{i_m} \dots i_l]}^{(k-1)} \otimes v_j \right. \\ \left. + \frac{2k}{h^\vee - 2k} \delta_{i_m, -j} \sum_{p \in J} v_p \otimes w_{[i_1 \dots \widehat{i_m} \dots i_l]}^{(k-1)} \otimes v_{-p} \right) \\ + (-)^l \sum_{p \in J} \left( v_j \otimes w_{[i_1 \dots i_{l-p}]}^{(k-1)} \otimes v_p - k v_p \otimes w_{[i_1 \dots i_{l-p}]}^{(k-1)} \otimes v_j \right. \\ \left. + \frac{2k}{h^\vee - 2k} v_p \otimes w_{[i_1 \dots i_l j]}^{(k-1)} \otimes v_{-p} \right). \end{aligned}$$

We prove the lemma by induction on  $k$ . Note that  $\widehat{S}_{1k+1} = \widehat{S}_{2k+1} R_{1k+1}^{(k)}$ , and assume the formula for  $k-1$ . After some calculation using the properties of  $w_{[i_1 \dots i_l]}^{(k)}$ , we get the desired formula.  $\square$

To complete the proof, it suffices to show

**Lemma A2.**

- (i)  $w_{[i_1 \dots i_l]}^{(k)} \in \text{Im } T^{(k)}$  for all  $i_1, \dots, i_l \in J$ ,
- (ii) The vectors  $\{w_{[i_1 \dots i_l]}^{(k)} \mid l = k, k-2, \dots, 0 \text{ or } 1, i_1 \prec i_2 \prec \dots \prec i_l\}$  are linearly independent.

*Proof.* We prove (i) by the induction on  $k$ . From the induction assumption we know that  $w_{[i_1 \dots i_{l-1}]}^{(k-1)} \in \text{Im } T^{(k-1)}$ . From (A2a) we can assume  $i_1, \dots, i_{l-1}$  are distinct. We divide the proof into the following cases:

- (a)  $i_m + i_l \neq 0$  for  $m = 1, \dots, l-1$ ,
- (b) there exists  $a$  ( $1 \leq a \leq l-1$ ) such that  $i_a + i_l = 0$ ,
- (c)  $l = 0$ .

Put

$$A = (-)^{k-1} (k-1)! \frac{h^\vee - k - l + 2}{h^\vee - 2k + 2}, \quad B = (k-1)! \frac{k - l + 2}{h^\vee - 2k + 2}.$$

For (a) see lemma A1. We have

$$\widehat{S}_{1k}(w_{[i_1 \dots i_{l-1}]}^{(k-1)} \otimes v_{i_l}) = A w_{[i_1 \dots i_l]}^{(k)} \in \widehat{S}_{1k}(\text{Im } T^{(k-1)} \otimes V) = \text{Im } T^{(k)}.$$

For (b) we similarly have

$$\begin{aligned}\widehat{S}_{1k}(w_{[i_1 \dots i_a \dots i_{l-1}]}^{(k-1)} \otimes v_{-i_a}) &= Aw_{[i_1 \dots i_a \dots i_{l-1} - i_a]}^{(k)} + Bw_{[i_1 \dots \widehat{i_a} \dots i_l]}^{(k)}, \\ \widehat{S}_{1k}(w_{[i_1 \dots -i_a \dots i_{l-1}]}^{(k-1)} \otimes v_{i_a}) &= Aw_{[i_1 \dots -i_a \dots i_{l-1} i_a]}^{(k)} + Bw_{[i_1 \dots \widehat{-i_a} \dots i_l]}^{(k)}.\end{aligned}$$

Recalling that  $w_{[i_1 \dots i_a \dots i_{l-1} - i_a]}^{(k)} = -w_{[i_1 \dots -i_a \dots i_{l-1} i_a]}^{(k)}$ , we get

$$w_{[i_1 \dots i_a \dots i_{l-1} - i_a]}^{(k)}, w_{[i_1 \dots \widehat{i_a} \dots i_{l-1}]}^{(k)} \in \text{Im } T^{(k)}.$$

For (c) apply (b) to the case  $l = 2$ .

We proceed to the proof of (ii). The proof is reduced to showing linear independence of the vectors with the same weight. So we are to show for  $\alpha_1, \dots, \alpha_l \in J$  ( $\alpha_1 \prec \dots \prec \alpha_l$ )

$$\sum_I^* a_I w_{[\alpha_1 \dots \alpha_l i_1 \dots i_m - i_m \dots - i_1]}^{(k)} = 0 \implies a_I = 0. \quad (\text{A3})$$

Here  $\sum^*$  is the summation on  $I = (i_1 \dots i_m)$  such that  $1 \leq i_1 < \dots < i_m \leq n$ ,  $i_p \neq \alpha_q$  for any  $(p, q)$ . We use induction on  $k$ . Since the case  $l = k$  is trivial, we assume  $l < k$ . Note the following simple fact:

$$\sum_{j \in J} v_j \otimes w_j = 0 \text{ for } w_j \in V^{\otimes(k-1)} \implies w_j = 0 \text{ for all } j.$$

Fix  $j > 0$  such that  $j \neq \pm \alpha_1, \dots, \pm \alpha_l$ . Look at the first component of the sum (A3) and use the above observation, then as coefficients of  $v_j$  and  $v_{-j}$  we have

$$\begin{aligned}0 &= \sum_I \theta(i_1 = j) a_{[j i_2 \dots i_m]} (-)^l w_{[\alpha_1 \dots \alpha_l i_2 \dots i_m - i_m \dots - i_2 - j]}^{(k-1)} \\ &\quad + \sum_I \theta(i_2 = j) a_{[i_1 j i_3 \dots i_m]} (-)^{l+1} w_{[\alpha_1 \dots \alpha_l i_1 i_3 \dots - i_3 - j - i_1]}^{(k-1)} \\ &\quad \vdots \\ &= \sum_I \theta(i_m = j) a_{[i_1 \dots i_{m-1} j]} (-)^{l+m-1} w_{[\alpha_1 \dots \alpha_l i_1 \dots i_{m-1} - j - i_{m-1} \dots - i_1]}^{(k-1)} \\ &\quad + \theta(m \neq \frac{k-l}{2}) \sum_I a_I (-)^l w_{[\alpha_1 \dots \alpha_l i_1 \dots i_m - i_m \dots - i_1 - j]}^{(k-1)}, \\ 0 &= \sum_I \theta(i_1 = j) a_{[j i_2 \dots i_m]} (-)^{l+1} w_{[\alpha_1 \dots \alpha_l j i_2 \dots i_m - i_m \dots - i_2]}^{(k-1)} \\ &\quad + \sum_I \theta(i_2 = j) a_{[i_1 j i_3 \dots i_m]} (-)^{l+2} w_{[\alpha_1 \dots \alpha_l i_1 j i_3 \dots - i_3 - i_1]}^{(k-1)} \\ &\quad \vdots \\ &= \sum_I \theta(i_m = j) a_{[i_1 \dots i_{m-1} j]} (-)^{l+m} w_{[\alpha_1 \dots \alpha_l i_1 \dots i_{m-1} j - i_{m-1} \dots - i_1]}^{(k-1)} \\ &\quad + \theta(m \neq \frac{k-l}{2}) \sum_I a_I (-)^l w_{[\alpha_1 \dots \alpha_l i_1 \dots i_m - i_m \dots - i_1 j]}^{(k-1)}\end{aligned}$$

From the induction assumption we have

$$a_{[i_1 \dots i_p j i_{p+1} \dots i_m]} \pm a_{[i_1 \dots i_m]} = 0 \quad \text{for all } I = (i_1 \dots i_m),$$

where  $p$  is given from  $i_p < j < i_{p+1}$ . This completes the proof.  $\square$

**A.3 Proof of (4.6).** In this subsection we use the following notations.

$$\begin{aligned} \Delta^{(k)'}(x) &= \sum x_{(k)} \otimes \dots \otimes x_{(1)} \quad \text{if } \Delta^{(k)}(x) = \sum x_{(1)} \otimes \dots \otimes x_{(k)}, \\ P^{(k)} &\text{ is the linear operator } P^{(k)}v_1 \otimes \dots \otimes v_k = v_k \otimes \dots \otimes v_1 \ (v_j \in V), \\ \check{T}^{(k)} &= P^{(k)}T^{(k)}. \end{aligned}$$

We prepare two lemmas. The first can be shown directly.

**Lemma A3.**

$$\begin{aligned} \Delta^{(k)} \circ a^{\pm 1} &= (a^{\pm 1} \otimes \dots \otimes a^{\pm 1}) \circ \Delta^{(k)'}, \\ \Delta^{(k)'} \circ a^{\pm 1} &= (a^{\pm 1} \otimes \dots \otimes a^{\pm 1}) \circ \Delta^{(k)}. \end{aligned}$$

**Lemma A4.**

- (1)  $\check{T}^{(k)}$  is an isomorphism on  $\text{Im } \check{T}^{(k)}$ .
- (2)  $V^{\otimes k} = \text{Im } \check{T}^{(k)} \oplus \text{Ker } T^{(k)}$ .

*Proof.* Since  $\text{Im } \check{T}^{(k)} \cong V^{(k)}$  as a  $U_q(\mathfrak{g})$ -module, we know from (4.5) that

$$\text{Im } \check{T}^{(k)} \cong V_{\bar{\Lambda}_k} \oplus V_{\bar{\Lambda}_{k-2}} \oplus \dots \oplus V_{\bar{\Lambda}_1} \text{ or } V_0 \quad \text{as a } U_q(\bar{\mathfrak{g}})\text{-module.}$$

Therefore we have

$$\check{T}^{(k)} = \lambda_k \text{id}_{V_{\bar{\Lambda}_k}} + \lambda_{k-2} \text{id}_{V_{\bar{\Lambda}_{k-2}}} + \dots$$

with some scalars  $\lambda_k, \lambda_{k-2}, \dots \neq 0$ . For (2) it suffices to show that  $\text{Im } \check{T}^{(k)} \cap \text{Ker } T^{(k)} = 0$ . Taking  $\check{T}^{(k)}v \in \text{Im } \check{T}^{(k)} \cap \text{Ker } T^{(k)}$ , we have  $(\check{T}^{(k)})^2v = 0$ . From (1) we get  $\check{T}^{(k)}v = 0$ .  $\square$

In what follows, we explicitly write the  $U_q(\mathfrak{g})$ -module structure in such a manner as  $(V_{z(-q)^{1-k}} \otimes \dots \otimes V_{z(-q)^{k-1}})^{*a^{\pm 1}}, V_{z(-q)^{1-k}} \otimes' \dots \otimes' V_{z(-q)^{k-1}}$ , etc. Here  $\otimes'$  signifies the use of the opposite coproduct  $\Delta'$ . Let us start by noting an isomorphism

$$\begin{aligned} V_z^{(k)} &\stackrel{\text{def}}{=} (V_{z(-q)^{1-k}} \otimes \dots \otimes V_{z(-q)^{k-1}}) / \text{Ker } T^{(k)} \\ &\xrightarrow{\sim} T^{(k)}(V_{z(-q)^{1-k}} \otimes \dots \otimes V_{z(-q)^{k-1}}). \end{aligned}$$

Consider the following sequence of homomorphisms:

$$(V_z^{(k)})^{*a^{\pm 1}} = ((V_{z(-q)^{1-k}} \otimes \cdots \otimes V_{z(-q)^{k-1}}) / \text{Ker } T^{(k)})^{*a^{\pm 1}} \\ \xrightarrow{\sim} \{f : V_{z(-q)^{1-k}} \otimes \cdots \otimes V_{z(-q)^{k-1}} \rightarrow \mathbf{Q}(q) \mid f : \text{Ker } T^{(k)} \rightarrow 0\} \quad (\text{A4})$$

$$\hookrightarrow (V_{z(-q)^{1-k}} \otimes \cdots \otimes V_{z(-q)^{k-1}})^{*a^{\pm 1}} \\ \xrightarrow{\sim} (V_{z(-q)^{1-k}})^{*a^{\pm 1}} \otimes' \cdots \otimes' (V_{z(-q)^{k-1}})^{*a^{\pm 1}} \quad (\text{A5})$$

$$\xrightarrow{\sim} V_{z\xi^{\mp 1}(-q)^{1-k}} \otimes' \cdots \otimes' V_{z\xi^{\mp 1}(-q)^{k-1}}. \quad (\text{A6})$$

The isomorphism (A5) is due to Lemma A3, and (A6) is given by  $C_{\pm}^{-1} \otimes \cdots \otimes C_{\pm}^{-1}$ , where  $C_{\pm} = C_{\pm}^{(1)}$  is defined in (3.10). The remaining thing is to show that the image of (A4) in (A6) is identified as  $T^{(k)}(V_{z\xi^{\mp 1}(-q)^{1-k}} \otimes \cdots \otimes V_{z\xi^{\mp 1}(-q)^{k-1}})$ . First we can show

$$\{f : V_{z(-q)^{1-k}} \otimes \cdots \otimes V_{z(-q)^{k-1}} \rightarrow \mathbf{Q}(q) \mid f : \text{Ker } T^{(k)} \rightarrow 0\} \\ \xrightarrow{\sim} T^{(k)*}(V_{z(-q)^{1-k}} \otimes' \cdots \otimes' V_{z(-q)^{k-1}})^{*a^{\pm 1}}, \\ f \mapsto \check{T}^{(k)*}f,$$

in the following manner. We can easily show that the map is an homomorphism. Noting Lemma A4(2), for  $g : V_{z(-q)^{1-k}} \otimes' \cdots \otimes' V_{z(-q)^{k-1}} \rightarrow \mathbf{Q}(q)$  we set

$$f(v) = \begin{cases} g(P^{(k)}v) & v \in \text{Im } \check{T}^{(k)} \\ 0 & v \in \text{Ker } T^{(k)}. \end{cases}$$

We can show  $\check{T}^{(k)*}f = T^{(k)*}g$ . Thus we have shown the surjectivity. The injectivity can also be shown using Lemma A4(2).

To finish we have to show that the image of  $T^{(k)*}(V_{z(-q)^{1-k}} \otimes' \cdots \otimes' V_{z(-q)^{k-1}})^{*a^{\pm 1}}$  by  $C_{\pm}^{-1} \otimes \cdots \otimes C_{\pm}^{-1}$  is  $T^{(k)}(V_{z\xi^{\mp 1}(-q)^{1-k}} \otimes \cdots \otimes V_{z\xi^{\mp 1}(-q)^{k-1}})$ , or

$$(C_{\pm}^{-1} \otimes \cdots \otimes C_{\pm}^{-1})T^{(k)t}(C_{\pm} \otimes \cdots \otimes C_{\pm}) = T^{(k)}.$$

This reduces to checking  $(C_{\pm}^{-1} \otimes C_{\pm}^{-1})\overline{R}(z)^t(C_{\pm} \otimes C_{\pm}) = \overline{R}(z)$ , which can be done by a direct calculation.

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